Admin. notes

- HW #9 to be posted later today (Last HW, due Friday 3/15)
- Will post two practice final exams

Last

- Rayleigh quotients
- Definite matrices (positive definite, positive semi-definite, negative definite, negative semi-definite)
- Positive definite (and Hermitian)
  - $x^*Ax \geq 0, \forall x$, and "" iff $x = 0$
  - $\lambda_1(A) \geq \ldots \geq \lambda_n(A) > 0$
  - $A$ has a unique Cholesky decomposition $A = LL^*$

- All leading principal minors of $A$ are positive
Positive semidefinite matrices (and Hermitian)

- $x^\top A x \geq 0, \forall x$
- $\lambda_1(A) \geq \ldots \geq \lambda_n(A) \geq 0$
- $A$ has a Cholesky decomposition $A = LL^\top$
  (not necessarily unique)
- All principal minors of $A$ are nonnegative

\[ A_{31,23} \]
\[ A_{41,33} \]
\[ \text{det} (A_s) \]

lower triangular
or nonnegative diagonal entries

$A_{31,23}$

$A_{41,33}$

when $A_s = [A_{ij}]_{i,j \in s}$ for some $s \subseteq [n]$
Let $A$ be positive semidefinite and Hermitian.

Goal: Find $L$ (lower triangular with nonnegative diagonal entries) such that

$$A = LL^*$$

Example:

$$A = \begin{bmatrix}
4 & 12 & -16 \\
12 & 37 & -43 \\
-16 & -43 & 98
\end{bmatrix} = \begin{bmatrix}
2 \\
6 & 1 \\
-8 & 5 & 3
\end{bmatrix} \begin{bmatrix}
2 & 6 & -8 \\
6 & 1 & 5 \\
-8 & 5 & 3
\end{bmatrix}$$

$$-43 = -8 \cdot 6 + L_{32} \cdot 1 + L_{33} \cdot 0$$

$$98 = 64 + 25 + L_{33}^2$$
We now know how to decompose a psd Hermitian $A$ as $A = BB^*$ (take $B$ as $L$ in Cholesky). Is this a unique way of writing $A = BB^*$? The answer is no. Let $A = U \Lambda U^*$, and $B = U \Lambda^{1/2} U^*$. Then $BB^* = B^2$

\[ BB^* = \begin{bmatrix} \lambda_1^{1/2} & \vdots & \lambda_n^{1/2} \\ \vdots & \ddots & \vdots \\ \lambda_1^{1/2} & \vdots & \lambda_n^{1/2} \end{bmatrix} \]

Let $\Lambda = \begin{bmatrix} \lambda_1 & \vdots & \lambda_n \end{bmatrix}$. Then

\[ BB^* = U \Lambda \Lambda^{1/2} U^* = A \]
Applicahn 1 (Probability of random vectors)

Let $x = (X_1, X_2, \ldots, X_n)$ be a random vector. Then the mean of $x$ is

$$E[x] = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix}$$

and the covariance matrix of $x$ is

$$K_x = \begin{bmatrix} \text{Cov}(X_i, X_j) \\ \vdots \\ \text{Cov}(X_n, X_n) \end{bmatrix}$$

where $\text{Cov}(X_i, X_j) = E[(X_i - E[X_i])(X_j - E[X_j])]$. 


Fact 1. For any random vector $x$, its covariance matrix $K_x$ is positive semi-definite and symmetric (Hermitian).

**Proof:** Write

$$K_x = E \left[ \begin{bmatrix} X_1 - \mathbb{E}X_1 \\ \vdots \\ X_n - \mathbb{E}X_n \end{bmatrix} \mathbb{E} \begin{bmatrix} X_1 - \mathbb{E}X_1 \\ \vdots \\ X_n - \mathbb{E}X_n \end{bmatrix} \right]$$

$$= E \left[ (X - \mathbb{E}X) (X - \mathbb{E}X)' \right].$$

Hence, any $a \in \mathbb{R}^n$,

$$a' K_x a = a' E \left[ (X - \mathbb{E}X) (X - \mathbb{E}X)' \right] a = E \left[ a' (X - \mathbb{E}X) (X - \mathbb{E}X)' a \right] = E \left[ (a' (X - \mathbb{E}X))^2 \right] \geq 0 \quad \forall a$$
Fact 2. Every symmetric (Hermitian) positive semidefinite matrix is the covariance matrix of some random vector.

Let $A \in \mathbb{S}^n$ be symmetric and $A = BB'$.

Let $x \sim N(0, I_n)$, i.e., $X_1, X_2, \ldots, X_n$ be i.i.d. $N(0, 1)$ (independent and identically distributed) Gaussian random variables.

Then consider $y = Bx$ as another random vector formed from $x$. The covariance matrix of $y$ is

$$K_y = E[(y - \mu_y)(y - \mu_y)'] = E[(Bx - E(Bx))(Bx - E(Bx))']$$

$$= B E[(x - \mu_x)(x - \mu_x)'] B'$$

Since $K_x = I_n$

$$= BB'.$$
Application 2. Ellipsoids

Let $B \in \mathbb{R}^{n \times n}$ be nonsingular, and $A = B'B > 0$.

$$B(E) = \{ Bx : x'Ax \leq 1 \}$$

$$= \{ z : (B'z)'A(B'z) \leq 1 \}$$

$$= \{ z : z'(B'B'B'z) \leq 1 \}$$

$$= \{ z : z'z \leq 1 \} = S$$