Solutions to Homework Set #6
(Prepared by TA Alankrita Bhatt)

1. Gershgorin circles. Let $v$ be an eigenvector of $A \in \mathbb{C}^{n \times n}$ associated with eigenvalue $\lambda$ such that $\|v\|_\infty = |v_i| = 1$.

(a) Show that

$$\lambda - A_{ii} = \sum_{j \neq i} A_{ij} v_j.$$  

(b) Let the Gershgorin circles of $A$ be defined as

$$G_i = \{ \xi \in \mathbb{C} : |\xi - A_{ii}| \leq \rho_i \}, \quad i = 1, 2, \ldots, n,$$

where the radius of the $i$-th circle centered at $A_{ii}$ is

$$\rho_i = \sum_{j \neq i} |A_{ij}|.$$

Show that all eigenvalues of $A$ are contained in the union of the Gershgorin circles.

**Solution:**

(a) Since $v$ is an eigenvector associated with eigenvalue $\lambda$, we have $Av = \lambda v$. In particular, equating the $i$-th element of $Av$ and $\lambda v$, we have $\lambda v_i = \sum_{j=1}^{n} A_{ij} v_j = v_i A_{ii} + \sum_{j \neq i} A_{ij} v_j$, which yields $(\lambda - A_{ii})v_i = \sum_{j \neq i} A_{ij} v_j$.

(b) From part (a), we have

$$|\lambda - A_{ii}| |v_i| = \left| \sum_{j \neq i} A_{ij} v_j \right| \leq \sum_{j \neq i} |A_{ij}| |v_j| \leq \sum_{j \neq i} |A_{ij}|.$$

Thus, $|\lambda - A_{ii}| \leq \rho_i$, or equivalently, $\lambda \in G_i$. Similarly, the other eigenvalues lie in one of the Gershgorin circles.

2. Diagonally dominated matrices. We say that $A \in \mathbb{C}^{n \times n}$ is diagonally dominated if

$$A_{ii} > \sum_{j \neq i} |A_{ij}|, \quad i = 1, 2, \ldots, n.$$
Show that a diagonally dominated matrix $A$ is nonsingular. (Hint: Use Gershgorin circles.)

**Solution:** Consider the Gershgorin circles $G_i$ from Problem #1. Since $0 \notin G_i$ for every $i$, no eigenvalue is 0. Hence, $A$ is nonsingular.

3. **Spectral mapping theorem.** Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic, namely, it can be expressed as a convergent power series

\[ f(\xi) = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \cdots \]

for every $\xi$ in some open set $S$. Suppose that $A \in \mathbb{C}^{n \times n}$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in S$ counting multiplicity.

(a) Suppose that $A$ is diagonalizable with eigenvalue decomposition $A = T \Lambda T^{-1}$. Show that

\[ f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots \]

has an eigenvalue decomposition $f(A) = T f(\Lambda) T^{-1}$.

(b) More generally, show that $f(A)$ has eigenvalues $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$ (counting multiplicity), be it diagonalizable or not.

**Solution:**

(a) Since $A^k = T \Lambda^k T^{-1}$, we have

\[
\begin{align*}
  f(A) &= \alpha_0 I + \alpha_1 T \Lambda T^{-1} + \alpha_2 T \Lambda^2 T^{-1} + \cdots \\
  &= T [\alpha_0 + \alpha_1 \Lambda + \alpha_2 \Lambda^2 + \cdots] T^{-1} \\
  &= T f(\Lambda) T^{-1},
\end{align*}
\]

where the convergence of the series in the powers of $\Lambda$ is guaranteed since $f$ is analytic.

(b) Suppose that $A = T J T^{-1}$ with Jordan canonical form $J$. We then have $f(A) = T (\alpha_0 + \alpha_1 J + \alpha_2 J^2 + \cdots) T^{-1}$. Since eigenvalues remain unchanged by a similarity transform, the eigenvalues of $f(A)$ are exactly the eigenvalues of $f(J)$. But since $f(J)$ is an upper-triangular matrix, the eigenvalues are given by the diagonal elements of $f(J)$. By the block diagonal structure of $J$, the $i$-th diagonal element of $f(J)$ is $\alpha_0 + \alpha_1 \lambda_i + \alpha_2 \lambda_i^2 + \cdots = f(\lambda_i)$, where the convergence of this series is guaranteed since $f$ is analytic. Thus, the eigenvalues of $f(A)$ are $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$.

4. **Nilpotent matrices.** We say that a square matrix $A$ is nilpotent if $A^k = 0$ for some $k \geq 1$. We define the smallest $k$ for which $A^k = 0$ to be its (nilpotent) index. For example,

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

is nilpotent of index 3.

(a) Show that every nilpotent matrix $A \in \mathbb{F}^{n \times n}$ has no nonzero eigenvalue and thus that its characteristic function is $\chi(\lambda) = \det(\lambda I - A) = \lambda^n$.
(b) Show that the index of a nilpotent matrix $A \in \mathbb{F}^{n \times n}$ is always $\leq n$.

(c) Suppose that $A \in \mathbb{F}^{n \times n}$ is nilpotent of index $n$. Show that if $A^{n-1}x \neq 0$, then $x, Ax, A^2x, \ldots, A^{n-1}x$ form a basis of $\mathbb{F}^n$.

(d) Continuing part (c), let

$$T = \begin{bmatrix} x & Ax & A^2x & \cdots & A^{n-1}x \end{bmatrix} \in \mathbb{F}^{n \times n}.$$  

Show that the similarity transformation of $A$ by $T$ is

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$  

Solution:

(a) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A$, a nilpotent matrix with index $k$. Since $\xi^k$ is an analytic function, by Problem 3 the eigenvalues of $A^k$ are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$. But $A^k = 0$ and its eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ must be all zero, which in turn implies that $\lambda_1, \lambda_2, \ldots, \lambda_n$ must be all zero. Therefore all $n$ eigenvalues of $A$ are zero, implying that $\chi(\lambda) = \lambda^n$.

(b) By the Cayley–Hamilton theorem, $\chi(A) = A^n = 0$. Thus, the index of a nilpotent matrix is at most $n$.

(c) To show that $x, Ax, A^2x, \ldots, A^{n-1}x$ are independent, consider some $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that $\alpha_0x + \alpha_1Ax + \alpha_2A^2x + \cdots + \alpha_{n-1}A^{n-1}x = 0$. Multiplying both sides by $A^{n-1}$ and since $A^k = 0$ for $k \geq n$, we have $\alpha_0A^{n-1}x = 0$, which implies that $\alpha_0 = 0$. Next, we multiply both sides by $A^{n-2}$ and use the fact that $\alpha_0 = 0$ to show $\alpha_1 = 0$. Continuing this way, we can show that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are all zero. Thus, $x, Ax, A^2x, \ldots, A^{n-1}x$ are $n$ independent vectors that form a basis for $\mathbb{F}^n$.

(d) Note that

$$AT = A \begin{bmatrix} x & Ax & A^2x & \cdots & A^{n-1}x \end{bmatrix} = \begin{bmatrix} Ax & A^2x & A^3x & \cdots & A^n x \\ Ax & A^2x & A^3x & \cdots & A^n x \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Ax & A^2x & A^3x & \cdots & 0 \end{bmatrix} = T \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$  

Thus,

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$  

5. Gram matrices. Let $V$ be an inner product space over $\mathbb{C}$ and $v_1, v_2, \ldots, v_n \in V$. The Gram matrix $G \in \mathbb{C}^{n \times n}$ associated with these vectors is defined by

$$G_{ij} = \langle v_i, v_j \rangle.$$
6. Properties of symmetric matrices. Let \( A = A^T \in \mathbb{R}^{n \times n} \) and \( B = B^T \in \mathbb{R}^{n \times n} \). Prove or provide a counterexample to each of the following statements.

(a) If \( A \succeq 0 \), then \( X^TAX \succeq 0 \) for every \( X \in \mathbb{R}^{n \times k} \).
(b) If \( A \succeq 0 \) and \( B \succeq 0 \), then \( \text{tr}(AB) \geq 0 \).
(c) If \( A \succeq 0 \), then \( A + B \succeq B \).
(d) If \( A \succeq B \), then \( -B \succeq -A \).
(e) If \( A \succeq I \), then \( I \succeq A^{-1} \).
(f) If \( A \succeq B > 0 \), then \( B^{-1} \succeq A^{-1} > 0 \).
(g) If \( A \succeq B \geq 0 \), then \( A^2 \succeq B^2 \).

Solution: All statements are true except (g).

(a) Given any \( y \in \mathbb{R}^n \), \( y^T (X^TAX)y = (Xy)^T A(Xy) \geq 0 \), since \( A \succeq 0 \). Thus, \( X^TAX \succeq 0 \). We can in fact show that if \( A \succeq 0 \) and \( X \) is full-rank and tall, then \( X^TAX \succeq 0 \). To see this, consider \( y^T X^TAXy \geq 0 \) with equality only if \( Xy = 0 \). But since the columns of \( X \) are linearly independent, \( Xy = 0 \) if and only if \( y = 0 \).

(b) Since \( A \) and \( B \) are symmetric and PSD we can use the result proved in Problem #5 to say that \( A = UU^T \) and \( B = VV^T \) for some \( U \) and \( V \). We then have \( \text{tr}(AB) = \text{tr}((UU^TV)V^T) = \text{tr}(V^T(UU^TV)) \). Since \( V^TUU^TV = (U^TV)^T(U^TV) \) is symmetric, positive semidefinite, its trace (which is the sum of the eigenvalues) is nonnegative.

(c) Since \( A = (A + B) - B \succeq 0 \), we have \( A + B \succeq B \).
(d) If \( A \succeq B \), we have \( A - B \succeq 0 \), which implies that \( -B - (-A) \succeq 0 \), and thus that \( -B \succeq -A \).
(e) Since $A$ is symmetric, we have $A = QΛQ^T$, where $QQ^T = I$. Thus, $A - I = Q(Λ - I)Q^T$, and the eigenvalues of $A - I$ are the eigenvalues of $A$ minus 1. Thus, if $A - I$ is positive semidefinite, every eigenvalue $λ$ of $A ≥ 1$. Now $A^{-1} = QΛ^{-1}Q^T$ with eigenvalues ≤ 1. Hence, $(I - A^{-1}) = Q(I - Λ^{-1})Q^T$ is positive semidefinite.

(f) Since $B = QΛQ^T ≻ 0$, $B^{1/2} = QΛ^{1/2}Q^T ≻ 0$. Then by part (a), $A - B ≥ 0$ implies $B^{-1/2}(A - B)B^{-1/2} = B^{-1/2}AB^{-1/2} - I ≥ 0$. Hence by part (e), $I - B^{1/2}A^{-1}B^{1/2} ≥ 0$. Finally, by part (a) once again, $B^{-1/2}(I - B^{1/2}A^{-1}B^{1/2})B^{-1/2} = B^{-1} - A^{-1} ≥ 0$.

(g) This need not be true. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $A ≥ B ≥ 0$. But $A^2 - B^2 = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ is indefinite.