Tiling a Rectangle by Squares

**Theorem.** A rectangle $R$ with side lengths 1 and $x$, where $x$ is irrational, cannot be “tiled” by finitely many squares (so that the squares have disjoint interiors and cover all of $R$).

**Proof.** For contradiction, let us assume that a tiling exists, consisting of squares $Q_1, Q_2, \ldots, Q_n$, and let $s_i$ be the side length of $Q_i$.

We need to consider the set $\mathbb{R}$ of all real numbers as a vector space over the field $\mathbb{Q}$ of rationals. This is a rather strange, infinite-dimensional vector space, but a very useful one.

Let $V \subseteq \mathbb{R}$ be the linear subspace generated by the numbers $x$ and $s_1, s_2, \ldots, s_n$, in other words, the set of all rational linear combinations of these numbers.

We define a linear mapping $f: V \to \mathbb{R}$ such that $f(1) = 1$ and $f(x) = -1$ (and otherwise arbitrarily). This is possible, because 1 and $x$ are linearly independent over $\mathbb{Q}$. Indeed, there is a basis $(b_1, b_2, \ldots, b_k)$ of $V$ with $b_1 = 1$ and $b_2 = x$, and we can set, e.g., $f(b_1) = 1$, $f(b_2) = -1$, $f(b_3) = \cdots = f(b_k) = 0$, and extend $f$ linearly on $V$.

For each rectangle $A$ with edges $a$ and $b$, where $a, b \in V$, we define a number $v(A) := f(a)f(b)$. 

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We claim that if the $1 \times x$ rectangle $R$ is tiled by the squares $Q_1, Q_2, \ldots, Q_n$, then $v(R) = \sum_{i=1}^{n} v(Q_i)$. This leads to a contradiction, since $v(R) = f(1)f(x) = -1$, while $v(Q_i) = f(s_i)^2 \geq 0$ for all $i$.

To check the claim just made, we extend the edges of all squares $Q_i$ of the hypothetical tiling across the whole of $R$, as is indicated in the picture:

This partitions $R$ into small rectangles, and using the linearity of $f$, it is easy to see that $v(R)$ equals to the sum of $v(B)$ over all these small rectangles $B$. Similarly $v(Q_i)$ equals the sum of $v(B)$ over all the small rectangles lying inside $Q_i$. Thus, $v(R) = \sum_{i=1}^{n} v(Q_i)$. \qed

Remark. It turns out that a rectangle can be tiled by squares if and only if the ratio of its sides is rational. Various other theorems about the impossibility of tilings can be proved by similar methods. For example, it is impossible to dissect the cube into finitely many convex pieces that can be rearranged so that they tile a regular tetrahedron.

Sources. The theorem is a special case of a result from


Unfortunately, so far I haven’t found the source of the above proof. Another very beautiful proof follows from a remarkable connection of square tilings to planar electrical networks: