1. **When is it true?** (5 points for each correct answer, -3 point for each wrong answer, 0 point for each blank) Fill in each blank with “always,” “sometimes,” or “never.” For example,

- A nonsingular matrix is **always** invertible.
- A square matrix is **sometimes** full-rank.
- A strictly tall matrix is **never** onto.

Here the matrix dimensions are such that each expression makes sense, but they are otherwise unspecified. Every vector and matrix has real entries.

(a) The union of two subspaces of \( \mathbb{R}^n \) is **sometimes** a subspace.

**Solution:** Consider two subspaces \( V \) and \( W \) of \( \mathbb{R}^n \) such that \( W \subseteq V \). In this case, \( V \cup W = V \) which is a subspace. On the other hand, consider the two subspaces \( V' \) and \( W' \) of \( \mathbb{R}^n \) spanned by the vectors \( v' = [1 \ 1 \ \cdots \ 1] \) (vector of all ones) and \( w' = [1 \ 1 \ \cdots -1] \) (vector zero at all positions except the last) respectively. Then, \( v' + w' = [1 \ 1 \ \cdots 0] \) which doesn’t belong to either of \( V' \) or \( W' \). Thus, \( V' \cup W' \) is not a subspace. In fact, it can be proved that for two subspaces \( V \) and \( W \), \( V \cup W \) is a subspace iff either \( V \subseteq W \) or \( W \subseteq V \).

(b) If \( AB = 0 \), then \( BA \) is **sometimes** a zero matrix.

**Solution:** Take \( A = \begin{bmatrix} 1 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \). Then \( AB = 0 \), but \( BA \neq 0 \). On the other hand, if \( A = 0 \) and \( B = 0 \), then both \( AB \) and \( BA \) are zero matrices.

(c) If \( \mathcal{R}(A) \subseteq \mathcal{N}(B^T) \), then \( BA \) is **always** a zero matrix.

**Solution:** Assume \( B \in \mathbb{R}^{m \times n} \) and \( A \in \mathbb{R}^{n \times k} \). Then, \( C := BA \in \mathbb{R}^{m \times k} \). If the columns of \( C \) are denoted as \( c_1, c_2, \cdots c_k \) and the columns of \( A \) as \( a_1, a_2, \cdots a_k \), then by the rules of matrix multiplication, \( c_i = Ba_i = 0 \) \( \forall i = \{1, 2, 3, \cdots k\} \) since \( a_i \in \mathcal{R}(A) \subseteq \mathcal{N}(B) \). Hence \( C = 0 \).

(d) If \( \mathcal{R}(A) \perp \mathcal{N}(B^T) \), then \( \text{rank}([A \ B]) \) is **always** equal to \( \text{rank}(B) \).

**Solution:** Recall that \( (\mathcal{N}(B^T))^\perp = \mathcal{R}(B) \). Since \( \mathcal{R}(A) \perp \mathcal{N}(B^T) \), \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \). Thus, \( \text{colspace}(A) \subseteq \text{colspace}(B) \). A basis for \( \mathcal{R}([A \ B]) \) is given by the union of the bases of \( \mathcal{R}(A) \) and \( \mathcal{R}(B) \), which by the previous observation is just \( \mathcal{R}(B) \). Thus, \( \text{rank}([A \ B]) = \dim(\mathcal{R}([A \ B])) = \text{rank}(B) \).

(e) The nullspace of \( \begin{bmatrix} A \\ A + B \end{bmatrix} \) is **always** equal to \( \mathcal{N}(A) \cap \mathcal{N}(B) \).

**Solution:** Let \( C = \begin{bmatrix} A \\ A + B \end{bmatrix} \). Then, if \( x \in \mathcal{N}(A) \cap \mathcal{N}(B) \), \( Cx = \begin{bmatrix} Ax \\ Ax + Bx \end{bmatrix} = 0 \), implying that \( x \in \mathcal{N}(C) \). Thus \( \mathcal{N}(A) \cap \mathcal{N}(B) \subseteq \mathcal{N}(C) \).
Also, if \( x \in \mathcal{N}, Cx = 0 \implies \begin{bmatrix} Ax \\ Ax + Bx \end{bmatrix} = 0. \) Clearly (by equating the respective blocks), this implies that \( Ax = 0 \) and \( Ax + Bx = 0, \) which in turn gives us \( Ax = 0 \) and \( Bx = 0. \) Thus, \( \mathcal{N}(C) \subseteq \mathcal{N}(A) \cap \mathcal{N}(B). \)

Putting these two together, we get \( \mathcal{N}(C) = \mathcal{N}(A) \cap \mathcal{N}(B). \)

(f) The nullspace of \( \begin{bmatrix} A \\ AB \end{bmatrix} \) is sometimes equal to \( \mathcal{N}(A) \cap \mathcal{N}(B). \)

**Solution:** Let \( A \in \mathbb{R}^{n \times k} \) and \( B \in \mathbb{R}^{k \times k}. \) If we take \( A = 0 \) and \( B = I, \) then \( \begin{bmatrix} A \\ AB \end{bmatrix} = 0 \) and \( \mathcal{N} \left( \begin{bmatrix} A \\ AB \end{bmatrix} \right) = \mathbb{R}^k \neq \mathcal{N}(A) \cap \mathcal{N}(B) = \mathbb{R}^k \cap \{0\} = \{0\}. \) However, if \( A \) is tall and full-rank and \( B = I, \) then \( \mathcal{N} \left( \begin{bmatrix} A \\ AB \end{bmatrix} \right) = \mathcal{N} \left( \begin{bmatrix} A \\ A \end{bmatrix} \right) = \{0\} = \mathcal{N}(A) \cap \mathcal{N}(B), \) since \( \mathcal{N}(A) = \mathcal{N}(B) = \{0\}. \)

(g) If \( A^T A \) is onto, then \( A \) is sometimes onto.

**Solution:** Let \( A \in \mathbb{R}^{m \times n}. \) Since \( \text{rank}(A) + \text{rank}(A^T) \leq \text{rank}(A^T A) \leq \text{rank}(A) \) and \( \text{rank}(A^T A) = n, \) the only possible value for \( \text{rank}(A) \) is \( n. \) Thus, \( \dim(\mathcal{R}(A)) = n. \) Now, if \( n < m, \) i.e. \( A \) is strictly tall, we have \( \mathcal{R}(A) \subset \mathbb{R}^m \) where the inclusion is strict. In this case, \( A \) is not onto. However, when \( m = n \) i.e. \( A \) is a square matrix, we have \( \mathcal{R}(A) \subset \mathbb{R}^n \) with dimension \( n. \) Thus, \( \mathcal{R}(A) = \mathbb{R}^n \) and \( A \) in this case is onto.

(h) If the matrix \( \begin{bmatrix} A \\ B \end{bmatrix} \) is onto, then \( A \) and \( B \) are always onto.

**Solution:** Let \( A \in \mathbb{R}^{m \times k} \) and \( B \in \mathbb{R}^{n \times k}. \) Consider \( \begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} Ax \\ Bx \end{bmatrix}. \) For \( \begin{bmatrix} Ax \\ Bx \end{bmatrix} \) to span \( \mathbb{R}^{m+n}, \) we need the columns of \( A \) to span \( \mathbb{R}^m, \) i.e., we need \( \mathcal{R}(A) = \mathbb{R}^m. \) This is because if for some vector \( v_1 \in \mathbb{R}^m, v_1 \notin \mathcal{R}(A), \) for any \( v_2 \in \mathbb{R}^n, \) \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) cannot belong to \( \mathcal{R} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right), \) contradicting our assumption that \( \begin{bmatrix} A \\ B \end{bmatrix} \) is onto. Thus, \( A \) needs to be onto. The proof that \( B \) is onto is similar.

(i) If \( A \) and \( B \) are onto, then \( \begin{bmatrix} A \\ 0 \\ C \\ B \end{bmatrix} \) is always onto.

**Solution:** To get \( \begin{bmatrix} A \\ C \\ 0 \\ B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \) (where \( v_1 \) and \( v_2 \) are arbitrary and have dimensions as appropriate), first note that \( \begin{bmatrix} A \\ 0 \\ C \\ B \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} Aa + Cb \\ Bb \end{bmatrix}. \) Choose \( b \) such that \( Bb = v_2. \) Such a \( b \) exists since \( B \) is onto. Next, choose an \( a \) such that \( Aa = v_1 - Cb. \) Again, since \( A \) is onto, such an \( a \) is guaranteed to exist. We have now shown that any arbitrary \( \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathcal{R} \left( \begin{bmatrix} A \\ 0 \\ C \\ B \end{bmatrix} \right). \) Thus, \( \begin{bmatrix} A \\ C \\ 0 \\ B \end{bmatrix} \) is onto if \( A \) and \( B \) are onto.

(j) The rank of \( AB \) is never greater than \( \text{rank}(A). \)

**Solution:** cf. Homework #2, Problem 5(b).
(k) The rank of $A + B$ is sometimes greater than rank($A$).

**Solution:** If $B = 0$, then $A + B = A$ and hence rank($A + B$) = rank($A$). On the other hand, consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A + B = I$, rank($A + B$) = 2 > rank($A$) = 1.

(l) If $A$ and $B$ are full-rank and tall, then $AB$ is always full-rank and tall.

**Solution:** Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$. Since $n \geq m$ and $m \geq k$, $n \geq k$ which implies that $AB \in \mathbb{R}^{m \times k}$ is tall. Also, $m + k - m \leq \text{rank}(AB) \leq \min(m, k) \implies \text{rank}(AB) = k$. Thus $AB$ is tall and full-rank.

(m) If $AB$ is full-rank, then $A$ and $B$ are sometimes full-rank.

**Solution:** Consider $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We observe that $AB$ is full-rank even though $A$ is not. On the other hand, if $A$ and $B$ are square matrices and $AB$ is full-rank, then $A$ and $B$ must be full-rank. Hence, $AB$ being full-rank doesn’t definitively indicate whether $A$ and $B$ are full-rank or not.

(n) If $A$ and $B$ are full-rank and $A^TB = 0$, then $[A \ B]$ is always full-rank.

**Solution:** Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times k}$. Since $A^TB = 0$, we have $\mathcal{R}(B) \subseteq \mathcal{N}(A^T) = \mathcal{R}(A)^\perp$, and thus the columns of $A$ and $B$ are orthogonal. This immediately implies that they are independent. Combined with the fact that $A$ and $B$ are full-rank (and hence have independent columns themselves), this implies that $[A \ B]$ is full-rank.

(o) If $A$ is full-rank and tall, then $\|Ax\| = 0$ always implies $x = 0$.

**Solution:** Since $A$ is tall and full-rank, by the rank-nullity theorem, we know that the dim($\mathcal{N}(A)$) = 0. Thus, $\mathcal{N}(A) = \{0\}$. Since for $\|Ax\| = 0$ we need $Ax = 0$ by definiteness property of the norm, $x = 0$.

(p) If $A$ is strictly fat, then $A^TA$ is never invertible.

**Solution:** Let $A \in \mathbb{R}^{m \times n}$. Then $A^TA \in \mathbb{R}^{n \times n}$. Since $A$ is strictly fat, rank($A$) $\leq \min(m, n) = m < n$. Now consider rank($A^TA$) $\leq$ rank($A$) $\leq n$. Hence, $A^TA$ cannot be full-rank and is hence not invertible.

(q) If $A$ has a rank decomposition $A = BC$, then $CC^T$ is always invertible.

**Solution:** Any nonzero matrix $A \in \mathbb{R}^{m \times n}$ ($\text{rank}(A) = r \geq 1$) can be decomposed as $A = BC$, where $B \in \mathbb{R}^{m \times r}$ is a full-rank tall matrix and $C \in \mathbb{R}^{r \times n}$ is a full-rank fat matrix. Now by Homework #3, Problem 7(f), $CC^T$ is invertible.

(r) If $A^2$ is invertible, then $A$ is always invertible.

**Solution:** We first note that for $A^2$ to be a valid matrix product, $A$ needs to be square. Assume $A \in \mathbb{R}^{n \times n}$. We then have rank($A^2$) $\leq$ rank($A$) $\leq \min(n, n) = n$. Since rank($A^2$) = $n$ for $A^2$ to be invertible, we must have rank($A$) = $n$. Thus, $A$ is full-rank and hence invertible.

(s) If the linear equation $y = Ax$ has a unique solution, then $A$ is sometimes square.

**Solution:** If $A = I$, then $Ax = y$ has a unique solution. However, $A$ need not necessarily be square for $Ax = y$ to have a solution. Consider $A = \begin{bmatrix} I \\ I \end{bmatrix}$ and $y = \begin{bmatrix} b \\ b \end{bmatrix}$ for some $b$. The
system of equations $Ax = y$ has a unique solution $x = b$ even though $A$ is not square.

(t) If the linear equation $y = Ax$ has multiple solutions, then $A$ is sometimes fat.

Solution: For the fat matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$ and for $y = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, the system of equations $Ax = y$ has infinitely many solutions. However, $A$ need not necessarily be fat for $Ax = y$ to have multiple solutions. Consider the tall matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The system of equations $Ax = y$ has infinitely many solutions of the form $x = (1, x_2)$ even though $A$ is square.

2. Permutation (30 points). Let $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$. Define a transformation $T : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ as $T(x) = (x_3, x_5, x_4, x_1, x_2)$.

(a) Find its inverse $T^{-1} : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ that satisfies $T^{-1}(T(x)) = T(T^{-1}(x)) = x$. For $y = (y_1, y_2, y_3, y_4, y_5)$, $T^{-1}(y) = (y_4, y_5, y_1, y_3, y_2)$.

Solution: $T^{-1}(y)$ is simply a permutation that moves the elements in $T(y)$ back to their original positions. In this case, $T^{-1}(y) = (y_4, y_5, y_1, y_3, y_2)$. We can verify that $T^{-1}(T(y)) = T(T^{-1}(y)) = y$.

(b) Find matrix representations of $T$ and $T^{-1}$ as $T(x) = Px$ and $T^{-1}(y) = Qy$:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Solution: By inspection.

(c) How are $P$ and $Q$ related? Answer: $P = Q^{-1} = Q^T$.

Solution: Since $T^{-1}(T(y)) = T(T^{-1}(y)) = y$, this implies that $PQy = QPy = y$ for all $y \in \mathbb{R}^5$. Hence, $PQ = QP = I$, or equivalently, $P$ and $Q$ are inverses of each other, i.e. $P = Q^{-1}$ and $Q = P^{-1}$. But since $P$ and $Q$ are orthogonal, $P = Q^T$ and $Q = P^T$ as well.

3. Kronecker product (30 points). For two matrices $A$ and $B$, define their Kronecker product $A \otimes B$ as the block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$
For example, if

\[
A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 & 1 \\ 3 & 1 & 2 \end{bmatrix},
\]

then

\[
A \otimes B = \begin{bmatrix}
2 & [1 & -2 & 1] \\
[3 & 1 & 2] & 3 & [1 & -2 & 1] \\
\end{bmatrix} = \begin{bmatrix}
2 & -4 & 2 & 3 & -6 & 3 \\
6 & 2 & 4 & 9 & 3 & 6 \\
-1 & 2 & -1 & 1 & -2 & 1 \\
-3 & -1 & -2 & 3 & 1 & 2
\end{bmatrix}.
\]

(a) \(I_m \otimes I_n = I_{mn}\).

**Solution:** Writing \(I_m \otimes I_n\) as

\[
I_m \otimes I_n = \begin{bmatrix}
I_n & 0 & \cdots & 0 \\
0 & I_n & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_n
\end{bmatrix}
\]

We can see from this representation that \(I_m \otimes I_n\) is a \(mn \times mn\) matrix with diagonal elements 1 and non-diagonal elements 0. Thus, \(I_m \otimes I_n = I_{mn}\).

(b) Express rank\((A \otimes B)\) in terms of rank\((A)\) and rank\((B)\):

\[
\text{rank}(A \otimes B) = \text{rank}(A) \cdot \text{rank}(B).
\]

**Solution:** Let rank\((A) = m\), rank\((B) = n\), and the basis vectors for \(\mathcal{R}(A)\) and \(\mathcal{R}(B)\) be \(v_1, v_2, v_3, \ldots, v_m\) and \(w_1, w_2, w_3, \ldots, w_n\) respectively. We note that the set of vectors \(v_i \otimes w_j, (i, j) \in [m] \times [n]\) form a basis set for \(\mathcal{R}(A \otimes B)\). Since there are \(mn\) of these vectors, rank\((A \otimes B) = mn = \text{rank}(A) \cdot \text{rank}(B)\).

(c) Assume that \(A\) and \(B\) are square and nonsingular. Express \((A \otimes B)^{-1}\) in terms of \(A^{-1}\) and \(B^{-1}\):

\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.
\]

**Solution:** Let \(A \in \mathbb{R}^{m \times m}\) and \(B \in \mathbb{R}^{n \times n}\). First of all, we note that \((A \otimes B)(C \otimes D) = AC \otimes BD\). Choosing \(C = A^{-1}\) and \(D = B^{-1}\), we get \((A \otimes B)(A^{-1} \otimes B^{-1}) = I_m \otimes I_n = I_{mn}\). Thus, \((A \otimes B)^{-1} = A^{-1} \otimes B^{-1}\).