1. Prove the Csiszár sum identity.

2. Mrs. Gerber’s Lemma. Let $H^{-1} : [0, 1] \rightarrow [0, 1/2]$ be the inverse of the binary entropy function.
   (a) (Difficult) Show that $H(H^{-1}(u) * p)$ is convex in $u$ for every $p \in [0, 1]$.
   (b) Use part (a) to prove the scalar MGL $H^{-1}(H(Y|U)) \geq H^{-1}(H(X|U)) * p$.
   (c) Use part (b) and induction to prove the vector MGL $H^{-1}\left(\frac{H(Y^n|U)}{n}\right) \geq H^{-1}\left(\frac{H(X^n|U)}{n}\right) * p$.

3. Conditional entropy power inequality. Let $X \sim f(x)$ and $Z \sim f(z)$ be independent random variables and $Y = X + Z$. Then by the EPI,
   $2^{2h(Y)} \geq 2^{2h(X)} + 2^{2h(Z)}$
   with equality iff both $X$ and $Z$ are Gaussian.
   (a) Show that $\log(2^x + 2^y)$ is convex in $(x, y)$.
   (b) Let $X^n$ and $Z^n$ be conditionally independent given an arbitrary random variable $U$, with conditional densities $f(x^n|u)$ and $f(z^n|u)$, respectively. Use part (a), the scalar EPI, and induction to prove the conditional EPI $2^{2h(Y^n|U)/n} \geq 2^{2h(X^n|U)/n} + 2^{2h(Z^n|U)/n}$.

4. Multiple access channel with degraded message sets. Consider a DM-MAC $p(y|x_1, x_2)$. Sender 1 encodes two independent messages $(M_0, M_1)$ uniformly distributed over $[1 : 2^{nR_0}] \times [1 : 2^{nR_1}]$, while sender 2 encodes only the message $M_0$. Thus, the common message $M_0$ is available to both senders, while the private message $M_1$ is available only to sender 1. The receiver $Y$ needs to decode both $M_0$ and $M_1$.
   (a) Show that the degraded message set capacity region is the set of rate pairs $(R_1, R_2)$ such that
      \[ R_1 \leq I(X_1; Y | X_2), \]
      \[ R_0 + R_1 \leq I(X_1, X_2; Y) \]
      for some $p(x_1, x_2)$.
   (b) Characterize the degraded message set capacity region of the AWGN-MAC with noise power 1, and channel gains $g_1$ and $g_2$, and input power constraints $P$ on both senders.
5. **Multiple access channel with common message.** Consider a DM-MAC \( p(y|x_1, x_2) \) with three messages \((M_0, M_1, M_2)\) uniformly distributed over \([1: 2^{nR_0}] \times [1: 2^{nR_1}] \times [1: 2^{nR_2}]\). Sender 1 encodes \((M_0, M_1)\) while sender 2 encodes \((M_0, M_2)\). Thus, the common message \(M_0\) is available to both senders, while the private messages \(M_1\) and \(M_2\) are available only to the respective senders. The receiver \(Y\) needs to decode all messages \(M_0, M_1, M_2\).

(a) Show that the three-message capacity region is the set of rate triples \((R_0, R_1, R_2)\) such that

\[
R_1 \leq I(X_1; Y|U, X_2), \\
R_2 \leq I(X_2; Y|U, X_1), \\
R_1 + R_2 \leq I(X_1, X_2; Y|U), \\
R_0 + R_1 + R_2 \leq I(X_1, X_2; Y)
\]

for some \(p(u)p(x_1|u)p(x_2|u)\).

(b) *(Very difficult)* Show that the three-message capacity region of the AWGN-MAC with noise power 1 and input power constraints \(P_1\) and \(P_2\) is the set of rate triples \((R_0, R_1, R_2)\) such that

\[
R_1 \leq C(\alpha_1 P_1), \\
R_2 \leq C(\alpha_2 P_2), \\
R_1 + R_2 \leq C(\alpha_1 P_1 + \alpha_2 P_2), \\
R_0 + R_1 + R_2 \leq C \left( P_1 + P_2 + 2\sqrt{\alpha_1 \alpha_2 P_1 P_2} \right)
\]

for some \(\alpha_1, \alpha_2 \in [0, 1]\).

Remark: The capacity region for the DM-MAC with common message is due to Slepian and Wolf (1973b). The converse for the Gaussian case is due to Bross, Lapidoth, and Wigger (2008).


7. **Successive cancellation vs. simultaneous decoding.** In Chapter 4 we found that for the DM-MAC, successive cancellation decoding with time sharing achieves the same rate region as simultaneous decoding. In this problem, we show that this is not the case for the interference channel.

Consider the AWGN-IC with SNRs \(S_1, S_2\) and INR \(I_1, I_2\).

(a) Write down the achievable rate region for successive cancellation decoding with Gaussian codes and no power control.

(b) Under what conditions is this region equal to the simultaneous decoding inner bound in Section 6.3?

(c) How much worse can successive cancellation decoding be than simultaneous decoding?

8. **Handoff.** Consider two symmetric AWGN-ICs, one with SNR \(S\) and INR \(I > S\), and the other with SNR \(I\) and INR \(S\). Thus, the second AWGN-IC is equivalent to the setting where the messages are sent to the other receivers in the first AWGN-IC. Which channel has a larger capacity region?

9. **Deterministic interference channels.** Consider the deterministic DM-IC model in Section 6.7. Suppose that \(X_1, X_2, T_1, T_2\) are binary and \(Y_1 = X_1 + T_2, Y_2 = X_2 + T_1\) are ternary.

(a) Let \(t_1(x_1) = x_1\) and \(t_2(x_2) = x_2\). Find the capacity region of the channel.

(b) Let \(t_1(x_1) = x_1\) and \(t_2(x_2) = 1 - x_2\). Find the capacity region of the channel.

10. **Strong deterministic interference channel.** Find the conditions on the functions of the class of deterministic interference channels in Section 6.7 under which the channel has strong interference.