1. Prove the achievability of the extended superposition coding inner bound (9.2) for the 3-receiver multilevel DM-BC.

   (a) Provide the details of the coding scheme outlined in the text.
   (b) Using the packing lemma and the mutual covering lemma, prove that the probability of error tends to zero as $n \to \infty$ if
      \[
      R_0 + R_{01} + R_{02} + \hat{R}_{11} < I(U_0, U_1; Y_1),
      \]
      \[
      R_0 + R_{01} + R_{02} + \hat{R}_{22} < I(U_0, U_2; Y_2),
      \]
      \[
      \hat{R}_{11} < I(U_1; Y_1 | U_0),
      \]
      \[
      \hat{R}_{22} < I(U_2; Y_2 | U_0),
      \]
      \[
      \hat{R}_{11} + \hat{R}_{22} - R_{11} - R_{22} > I(U_1; U_2 | U_0).
      \]
   (c) Establish the inner bound by using Fourier–Motzkin elimination to reduce the set of inequalities in part (b).
   (d) Show that the inner bound is convex.

3. Multiple-antenna Gaussian multiple access channel. Consider a two-user Gaussian multiple access channel with channel output $Y = (Y_1, Y_2)$ given by
   \[
   Y_1 = X_1 + Z_1
   \]
   \[
   Y_2 = X_1 + X_2 + Z_2,
   \]
   where channel inputs $X_1$ and $X_2$ from each user are both subject to power constraint $P$, and the zero-mean unit-variance Gaussian noises $Z_1$ and $Z_2$ are independent of each other and channel inputs.
   (a) Find the capacity region.
   (b) Find the time-division region (with power control). Is it possible to achieve any point on the boundary of the capacity region (except for the end points)?

4. Sato’s outer bound for the AWGN-IC. Show that if a rate pair $(R_1, R_2)$ is achievable for the AWGN-IC, then it must satisfy the conditions
   \[
   R_1 \leq C(S_1),
   \]
   \[
   R_2 \leq C(S_2),
   \]
   \[
   R_1 + R_2 \leq \frac{1}{2} \log(|PG^t| + 1),
   \]
   where
   \[
   G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}
   \]
is the channel gain matrix. Note that by Sato’s technique, the third inequality can be further tightened to obtain

\[ R_1 + R_2 \leq \min_{K_Z} \frac{1}{2} \log \left( \frac{|P G G^T + K_Z|}{|K_Z|} \right), \]

where

\[ K_Z = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \]

is the noise covariance.

Remark: The sum-rate bound is convex in \( K_Z \) (Kim and Kim 2006) and can be efficiently calculated even for a large number of sender–receiver pairs.