Homework Set #5
Due: Tuesday, February 26, 2008

1. Fourier–Motzkin elimination. Suppose \((R_{01}, R_{02}, R_{11}, R_{22})\) satisfies

\[
\begin{align*}
R_{11} &\leq I_1, \\
R_{11} + R_{01} &\leq I_2, \\
R_{11} + R_{02} &\leq I_3, \\
R_{11} + R_{01} + R_{02} &\leq I_4, \\
R_{22} &\leq I_5, \\
R_{22} + R_{02} &\leq I_6, \\
R_{22} + R_{01} &\leq I_7, \\
R_{22} + R_{02} + R_{01} &\leq I_8.
\end{align*}
\]

Show that \(R_1 = R_{01} + R_{11}\) and \(R_2 = R_{02} + R_{22}\) should satisfy

\[
\begin{align*}
R_1 &\leq I_2, \\
R_2 &\leq I_6, \\
R_1 + R_2 &\leq I_1 + I_8, \\
R_1 + R_2 &\leq I_3 + I_7, \\
R_1 + R_2 &\leq I_4 + I_5, \\
2R_1 + R_2 &\leq I_1 + I_4 + I_7, \\
R_1 + 2R_2 &\leq I_3 + I_5 + I_8.
\end{align*}
\]

2. Compound channel. A compound channel consists of a set of discrete memoryless channels \((\mathcal{X}, p_s(y|x), \mathcal{Y}), s \in \mathcal{S}\), with the same input and output alphabets but different conditional pmfs. It models a realistic situation where one knows only that the actual channel is one of several possible channels. Note that there is only one actual channel and it is used in all transmissions. A \((2^nR, n)\) code for the compound channel is defined in the same way as for the DMC (see lecture notes). The average probability of error is defined as

\[
P_e^{(n)} = \sup_s P \left\{ \hat{W} \neq W, \ s \ is \ the\ actual\ channel \right\}.
\]

A rate \(R\) is achievable if there exists a sequence of \((2^nR, n)\) codes with \(P_e^{(n)} \to 0\). The capacity \(C\) of the compound channel is the supremum over the set of achievable rates.
The capacity of the compound channel is given by

\[ C = \max_{p(x)} \inf_{s} I(X; Y_{s}), \]

where \( Y_{s} \mid \{X = x\} \sim p_{s}(y|x) \) for \( s \in \mathcal{S} \).

(a) Let \( C_{s} \) be the capacity of channel \( s \in \mathcal{S} \). Which of the following statements holds in general? Justify your answer.

i. \( C \geq \inf_{s} C_{s} \).

ii. \( C = \inf_{s} C_{s} \).

iii. \( C \leq C_{s} \).

(b) Compute the capacity of the binary symmetric compound channel with cross-over probability \( p \in [0, 0.11] \).

(c) Assuming the decoder knows the actual channel, but the code designer and encoder do not, prove achievability using random coding and joint typicality decoding.

(d) Prove the weak converse, that is, given any sequence of \( (2^{nR}, n) \) codes with \( P_{e}^{(n)} \to 0 \), \( R \leq C \).

Remark: Achievability is proved in the books by Wolfowitz and Csiszar and Korner without the assumption that the decoder knows the actual channel. This assumption is, however, not too unrealistic as the decoder can estimate the channel using a relatively short “test sequence.”

3. List codes. A \( (2^{nR}, 2^{nL}, n) \) list code for a DMC \( (\mathcal{X}, p(y|x), \mathcal{Y}) \) with capacity \( C \) consists of an encoder that assigns a codeword \( x_{n}(w) \) to each message \( w \in [2^{nR}] \) and a decoder that upon receiving \( y_{n} \) tries to find a list of codewords \( \mathcal{L}(y_{n}) \subset [2^{nR}] \) of size \( |\mathcal{L}| \leq 2^{nL} \) that contains the transmitted message. An error occurs if the list does not contain the transmitted message, i.e., \( P_{e}^{(n)} = P\{W \notin \mathcal{L}(Y^{n})\} \). An \( (R, L) \) pair is said to be achievable if there exists a sequence of \( (2^{nR}, 2^{nL}, n) \) list codes with \( P_{e}^{(n)} \to 0 \).

(a) Using random coding and joint typicality decoding, show that any \( (R, L) \) is achievable, provided \( R < C + L \).

(b) Show that given any sequence of \( (2^{nR}, 2^{nL}, n) \) list codes with \( P_{e}^{(n)} \to 0 \), \( R \leq C + L \).

(Hint: You will need to develop a modified Fano’s inequality.)