Source Coding for Multiple Descriptions
II: A Binary Source

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A uniformly distributed (iid) binary source is encoded into two binary data streams at rates \( R_1 \) and \( R_2 \), respectively. These sequences are such that by observing either one separately, a decoder can recover a good approximation of the source (at average error rates \( D_1, D_2 \), respectively), and by observing both sequences, a decoder can obtain a better approximation of the source (at average error rate \( D_0 \)). In this paper a “converse” theorem is established on the set of achievable quintuples \( (R_1, R_2, D_0, D_1, D_2) \). For the special case \( R_1 = R_2 = 1/2, D_0 = 0, \) and \( D_1 = D_2 = D \), our result implies that \( D \geq 1/5 \).

I. INTRODUCTION

Let \( \{X_k\}_{k=1}^{\infty} \) be a sequence of independent drawings of the binary random variable \( X \), where \( \Pr(X = 0) = \Pr(X = 1) = 1/2 \). Assume that this sequence appears at a rate of 1 symbol per second as the output of a data source. (Refer to Fig. 1.) An encoder observes this sequence and emits two binary sequences at rates \( R_1, R_2 \leq 1 \). These sequences are such that by observing either one, a decoder can recover a good approximation to the source output, and by observing both sequences, a decoder can obtain a better approximation to the source output. Letting \( D_1, D_2, \) and \( D_0 \) be the error rates which result when the streams at rate \( R_1 \), rate \( R_2 \), and both streams are used by a decoder, respectively, our problem is to determine (in the usual Shannon sense) the set of achievable quintuples \( (R_1, R_2, D_0, D_1, D_2) \). Our main result is a “converse” theorem which gives a necessary condition on the achiev-
able quintuples. This paper extends a previous one on the same subject. This paper, however, is self-contained.

This problem is an idealization of the situation in which it is desired to
(i) send information over two separate channels, as in a packet communication network, and
(ii) recover as much of the original information as possible, should one of the channels break down.

To fix ideas, let us say that \( R_1 = R_2 = 1/2, D_0 = 0, \text{ and } D_1 = D_2 = D. \) Thus, the source sequence at rate 1 is to be encoded into two sequences of rate 1/2 each, such that the original sequence can be recovered from these two encoded sequences with approximately zero error rate (i.e. \( D_0 = 0 \)). Our question then becomes: How well can we reconstruct the source sequence from one of the encoded streams—that is, what is the minimum \( D \)? A simple-minded approach would be to let the encoded streams consist of alternate source symbols, which will allow \( D_0 = 0 \). In this case, \( D = 1/4 \), since by observing every other source symbol a decoder will make an error half the time on the missing symbol. Is it possible to do better? El Gamal and Cover have looked at this problem and have a theorem which can be used to show that we can make \( D = (\sqrt{2} - 1)/2 \approx 0.207 \). In a previous paper it was shown that (with \( R_1 = R_2 = 1/2, D_0 = 0 \)) \( D \geq 1/5 = 0.200 \). The exact determination of the best \( D \) remains an open problem.*

II. FORMAL STATEMENT OF PROBLEM AND RESULTS

Let \( B = \{0, 1\} \), and let \( d_H(x, y) \), \( x, y \in B^n \), be the Hamming distance between the binary \( N \)-vectors \( x, y \); i.e., \( d_H(x, y) \) is the number of positions in which \( x \) and \( y \) do not agree. A code with parameters

\( (N, M_1, M_2, D_0, D_1, D_2) \) is a quintuple of real numbers where,

\[
\begin{align*}
&f_a: B^N \to \{1, \ldots, M_a\}, a = 1, 2, \ldots \\
g_a: 1, 2, \ldots, M_a \to B^N, a = 1, 2, \ldots \\
g_0: 1, 2, \ldots, M_1 \times \{(1, 2, \ldots)\} 
\end{align*}
\]

The source output is a random vector \( X \) uniform in \( B^n \).

Define

\[
Y = g_1 \circ f_1(X),
Z = g_2 \circ f_2(X),
\]

and

\[
\hat{X} = g_0[f_1(X), f_2(X)].
\]

Then the average error rates are

\[
\begin{align*}
D_1 &= \frac{1}{N} \text{Ed}_H(X, Y) \\
D_2 &= \frac{1}{N} \text{Ed}_H(X, Z) \\
D_0 &= \frac{1}{N} \text{Ed}_H(X, \hat{X})
\end{align*}
\]

We say that a quintuple \( (R_1, R_2, d_0, d_1, d_2) \) is arbitrary \( \epsilon > 0 \), there exists, for \( N \) sufficiently large, \( (N, M_1, M_2, D_0, D_1, D_2) \), where \( M_a \leq d_a + \epsilon, a = 0, 1, 2 \). The relationship of the system of Fig. 1 should be clear. Our problem is to find the set of achievable quintuples, and our next theorem.

Before stating our result, let us take a moment to review the theorem by El Gamal and Cover as it stands.

Theorem 1: The quintuple \( (R_1, R_2, d_0, d_1, d_2) \) exists a quadruple of random variables \( X, Y, Z, \hat{X} \) in \( B \), such that \( \Pr(X = 0) = \Pr(X = 1) = 1/2 \), where

\[
\begin{align*}
\text{Ed}_H(X, \hat{X}) &\leq d_0, \\
\text{Ed}_H(X, Y) &\leq d_1, \\
\text{Ed}_H(X, Z) &\leq d_2,
\end{align*}
\]

and

\[
R_1 \geq I(X; Y),
\]

* In Ref. 3, Witsenhausen proved a closely related result which encourages the conjecture that \( D = 0.207 \) is, in fact, the best possible.
For extends a previous one on the same subject and is self-contained.

Consider the situation illustrated in Fig. 1, where two separate channels, as in a packet network, are used to transmit the original information as possible, should errors occur. The rate 1 is to be encoded into two sequences of rate 1/2. At the receiver, the original sequence can be recovered from errors with approximately zero error rate (i.e., approximately 1/6). How well can we reconstruct the original frames of the encoded stream—that is, what is the error rate? The approach would be to let the encoded sequence correspond to the original symbol, which will allow errors to occur. By observing every other symbol and halving the time on the missing symbol, it would be possible to reconstruct the original symbol. For most error-prone channels, a simple solution is to make use of the error correction code and observe every other symbol. This basic idea is well known and has been used extensively in practice.

PROBLEM AND RESULTS

Let \((x, y)\), \(x, y \in \mathbb{B}^N\), be the Hamming distance function for vectors \(x, y\); i.e., \(d_H(x, y)\) is the number of positions where \(x\) and \(y\) do not agree. A code with parameters \((N, M_1, M_2, D_0, D_1, D_2)\) is a quintuple of mappings \((f_1, f_2, g_0, g_1, g_2)\) where,

\[
\begin{align*}
  f_2 : \mathbb{B}^N &\to \{1, \ldots, M_2\}, \quad \alpha = 1, 2 \quad (1a) \\
  g_1 : \{1, 2, \ldots, M_1\} &\to \mathbb{B}^N, \quad \alpha = 1, 2 \quad (1b) \\
  g_0 : \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\} &\to \mathbb{B}^N \quad (1c)
\end{align*}
\]

The source output is a random vector \(X\) uniformly distributed on \(\mathbb{B}^N\).

Define

\[
\begin{align*}
  Y &= g_1 f_1(X), \\
  Z &= g_2 f_2(X),
\end{align*}
\]

and

\[
\tilde{X} = g_0[f_1(X), f_2(X)].
\]

Then the average error rates are

\[
\begin{align*}
  D_1 &= \frac{1}{N} E_{d_H}(X, Y), \\
  D_2 &= \frac{1}{N} E_{d_H}(X, Z), \\
  D_0 &= \frac{1}{N} E_{d_H}(X, \tilde{X}).
\end{align*}
\]

We say that a quintuple \((R_1, R_2, d_0, d_1, d_2)\) is achievable if, for arbitrary \(\epsilon > 0\), there exists, for \(N\) sufficiently large, a code with parameters \((N, M_1, M_2, D_0, D_1, D_2)\), where \(M_\alpha \leq 2^{(R_\alpha + \epsilon)N}, \quad \alpha = 1, 2, \) and \(d_\alpha \leq d_\alpha + \epsilon, \quad \alpha = 0, 1, 2\). The relationship of this formalism to the system of Fig. 1 should be clear. Our problem is the determination of the set of achievable quintuples, and our main result is a converse theorem.

Before stating our result, let us take a moment to state a positive theorem by El Gamal and Cover as it specializes to our problem.

**Theorem 1:** The quintuple \((R_1, R_2, d_0, d_1, d_2)\) is achievable if there exists a quadruple of random variables \(X, \tilde{X}, Y, Z\), which take values in \(B\), such that \(\Pr(X = 0) = \Pr(X = 1) = 1/2\), and

\[
\begin{align*}
  E_{d_H}(X, \tilde{X}) &\leq d_0, \\
  E_{d_H}(X, Y) &\leq d_1, \\
  E_{d_H}(X, Z) &\leq d_2,
\end{align*}
\]

and

\[
R_1 \geq I(X; Y)
\]

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\[ R_2 \geq I(X; Z), \]  
\[ R_1 + R_2 \geq I(X; \bar{X}, Y, Z) + I(Y; Z), \]  
where \( I(\cdot; \cdot) \) is the usual Shannon information.

For the special case of \( R_1 = R_2 = 1/2, d_0 = 0 \), it can be shown that \( d_1 = d_2 = (\sqrt{2} - 1)/2 \approx 0.207 \) is achievable.

We now state our converse result.

**Theorem 2:** If \( (R_1, R_2, d_0, d_1, d_2) \) is achievable, then in all cases

\[ R_1 + R_2 \geq 1 - h(d_0), \]  

furthermore, if \( 2d_1 + d_2 \leq 1 \), then

\[ R_1 + R_2 \geq 2 - h(d_0) - h \left( 2d_1 + d_2 - \frac{2d_1^2}{1 - d_1} \right), \]  

and if \( d_1 + 2d_2 \leq 1 \), then

\[ R_1 + R_2 \geq 2 - h(d_0) - h \left( d_1 + 2d_2 - \frac{2d_2^2}{1 - d_1} \right), \]  

where

\[ h(\lambda) = \begin{cases} 
0, & \lambda = 0, \\
-\lambda \log_2 \lambda - (1 - \lambda) \log_2 (1 - \lambda), & 0 < \lambda \leq 1/2, \\
1, & \lambda \geq 1/2.
\end{cases} \]

All logarithms in this paper are taken to the base 2. As (6a) is obvious, and (6c) follows from (6b) by symmetry, we need only prove (6b).

In the special case of \( R_1 = R_2 = 1/2, d_0 = 0 \), and \( d_1 = d_2 = d \), inequality (6b) implies that

\[ h \left( 3d - \frac{2d^2}{1 - d} \right) \geq 1, \]

or

\[ 3d - \frac{2d^2}{1 - d} = \frac{3d(1 - d) - 2d^2}{1 - d} \geq 1, \]

which implies that \( d \geq 1/5 = 0.200 \).

**III. PROOF OF THEOREM 2**

We start from the standard identity

\[ I(U_1; U_2 | U_3) = I(U_1; U_2) + I(U_1; U_3 | U_2), \]  

for arbitrary random variables \( U_1, U_2, U_3 \). We say that \( U_1, U_2, U_3 \) is a "Markov chain" if \( U_1, U_3 \) are conditionally independent given \( U_2 \); i.e., \( U_3 \) depends on \( U_1, U_2 \) only through \( U_2 \). If \( U_1, U_2, U_3 \) is a Markov chain then \( I(U_1; U_3 | U_2) = 0 \), and from (7)

\[ I(U_1; U_2 U_3) = I(U_1; U_2) \]

Note that the hypothesis for (8) holds whenever the sequence \( \{U_n\} \) is a Markov chain if, for all \( n \)

\[ \cdots U_{n-2} U_{n-1} U_n, U_{n+1}, \ldots \]

is a Markov chain.

Let us now suppose that we are given a collection of parameters \( \{N, M_1, M_2, D_0, D_1, D_2\} \). We can show

\[
\log M_1 + \log M_2 \geq H(f_1(X)) + H(f_2(X)) + \frac{1}{2} I(\bar{X}; f_1(X), f_2(X)) \geq I(f_1(X); f_2(X)) + \frac{1}{2} I(\bar{X}; f_1(X), f_2(X)) + 1,
\]

where (9) follows from the fact that \( f_1(X) \) takes on at most \( M_1 \) values, (10) holds because the pair \( f_1(X), \bar{X} \) is independent, (11) holds because \( \bar{X}, Y \) and \( Z \) depend on different variables \( f_1(X) \) and \( f_2(X) \), and (12) follows from the fact that \( f_1(X) \) and \( f_2(X) \) are independent and identically distributed.

Now (11) is getting close to (5c) if we define

\[ A_k = \{X \colon f_1(X) = k\} \]

Let the cardinality of \( A_k \) be \( \mu_k \). Let the random variable \( Y \) be uniformly distributed over \( A_k \) and its conditional distribution given \( X \):

\[ \Pr(V = v | X = x) = \left[ \frac{\mu_k}{\mu} \right]^{-1}, \]

Thus, given \( X \in A_k \), \( V \) is uniformly distributed and the unconditional distribution for \( V \) is

\[ \Pr(V = v) = 2^{-n}, \quad v \in \{1, \ldots, 2^n\}, \]

and the components of \( V \) are independent, and \( Z, f_2(X), X, f_1(X) \) is a Markov chain, so that

\[ \Pr(V = v) | f_1(X) = k \]

This is valid for any distribution of \( X \).
I(X; Z), \quad (5b)
\[ I(X; \hat{X}, Y, Z) + I(Y; Z), \quad (5c) \]

Shannon information.
\[ R_2 = 1/2, \; d_0 = 0, \text{it can be shown that} \]
\[ \gamma \text{ is achievable.} \]

This result.
\[ + R_2 \geq 1 - h(d_0), \quad (6a) \]

\[ a(d_0) - h \left( 2d_1 + d_2 - \frac{2d_1^2}{1 - d_2} \right), \quad (6b) \]

\[ a(d_0) - h \left( d_1 + 2d_2 - \frac{2d_2^2}{1 - d_1} \right), \quad (6c) \]

\[ (1 - \lambda) \log_2 (1 - \lambda), \quad 0 < \lambda \leq 1/2. \]
\[ \lambda = 1/2. \]

are taken to the base 2. As (6a) is obvious, by symmetry, we need only prove (6b).
\[ R_2 = 1/2, \; d_0 = 0, \; d_1 = d_2 = d, \]
\[ 3d_1 - d_2 \geq 1, \]
\[ (1 - d) \geq 1/2, \]
\[ = 0.200. \]

Third identity
\[ I(U_1; U_2) + I(U_3; U_5|U_2), \quad (7) \]

uses \( U_1, U_2, U_3 \). We say that \( U_1, U_2, U_3 \) is a
are conditionally independent given \( U_2 \); i.e., through \( U_2 \). If \( U_1, U_2, U_3 \) is a Markov chain
from (7)

\[ I(U_1; U_2) \leq I(U_1; U_2 U_3) = I(U_1; U_2). \quad (8) \]

Note that the hypothesis for (8) holds when \( U_3 \) is a function of \( U_2 \). A sequence \( \{U_n\} \) is a Markov chain if, for all \( n \),
\[ \ldots U_{n-2}, U_{n-1}, U_n, (U_{n+1}, U_{n+2}, \ldots) \]
is a Markov chain.

Let us now suppose that we are given a code \( f_1, f_2, g_0, g_1, g_2 \) with parameters \( (N, M_1, M_2, D_0, D_1, D_2) \). We can write
\[ \log M_1 + \log M_2 \geq H(f_1(X)) + H(f_2(X)) \]
\[ = I(f_1(X); f_2(X)) + H(f_1(X) f_2(X)) \]
\[ = I(f_1(X); f_2(X)) + I(X; f_1(X) f_2(X)) \]
\[ \geq I(f_1(X); f_2(X)) + I(X; \hat{X}, Y, Z), \quad (9) \]

where (9) follows from the fact that \( f_i(X) \) takes its values in a set of cardinality \( M_i \), (10) holds because the pair \( f_i(X) f_2(X) \) is determined by \( X \) and (11) holds because \( \hat{X}, Y, \) and \( Z \) depend on \( X \) only through \( f_i(X) f_2(X) \) so that (8) applies.

Now (11) is getting close to (5c) in the direct theorem. In fact, using (8) twice, we can underbound \( I[f_i(X); f_2(X)] \) by \( I(Y; Z) \). Now the components of the source vector \( X \) are independent, and if the components of either \( Y \) or \( Z \) were also independent, we could make use of standard techniques to establish the necessity of (5c). But alas, we cannot assume that either of \( \{Y_n\} \) nor the \( \{Z_n\} \) are independent, so that another tactic is required. The key idea is the definition of another random vector \( V = (V_1, \ldots, V_n) \) the components of which are in fact independent.

For \( 1 \leq k \leq M_1 \), define the set
\[ A_k = \{x; f_i(x) = k\} = f_i^{-1}(k). \quad (12) \]

Let the cardinality of \( A_k \) be \( \mu_k \). Let the random vector \( V \) be defined by its conditional distribution given \( X \):
\[ \Pr(V = v|X = x) = \begin{cases} [f_i(x)]^{-1}, & v \in A_i(x), \\ 0, & \text{otherwise}. \end{cases} \quad (13) \]

Thus, given \( X \in A_k \), \( V \) is uniformly distributed on \( A_k \). It follows that the unconditional distribution for \( V \) is
\[ \Pr(V = v) = 2^{-n}, \quad v \in B^n, \]

and the components of \( V \) are independent, as desired.* Furthermore, \( Z, f_2(X), X, f_1(X), V \) is a Markov chain, so that, using (8),

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*In effect, \( V \) is obtained from \( f_i(X) \) by a channel with transition probabilities \( \Pr(X = x|f_i(X) = k) \) so that the distribution of \( V \) is the same as that of \( X \), hence, iid. This is valid for any distribution of \( X \).
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\[ I(f_1(X); f_2(X)) = I(V, f_1(X); f_2(X), Z) \geq I(V; Z). \]  
\[ (14) \]

Combining (11) and (14), we obtain
\[ \frac{1}{N} \log M_1 + \frac{1}{N} \log M_2 \geq \frac{1}{N} I(V; Z) + \frac{1}{N} I(X; \hat{X}, Y, Z) \]
\[ \geq \frac{1}{N} I(V; Z) + \frac{1}{N} I(X; \hat{X}) \]
\[ \geq 2 - h \left( \frac{Ed_H(V; Z)}{N} \right) - h \left[ \frac{Ed_H(X, \hat{X})}{N} \right] \]
\[ = 2 - h(\Delta) - h(D_0), \]
\[ (15a) \]

where
\[ \Delta = \frac{Ed_H(V; Z)}{N}. \]
\[ (15b) \]

Step (1) follows from the "rate-distortion bound" which states that if \( U \) is a random vector uniformly distributed in \( B^N \) (as are \( V \) and \( X \)), and \( \hat{U} \) is an arbitrary binary random vector, then \( I(U; \hat{U}) \geq 1 - h\left[ \frac{1}{N} Ed_H(U, \hat{U}) \right] \). (See Ref. 4.)

We will now obtain an upper bound on \( \Delta \) in terms of \( D_1 \) and \( D_2 \). As a "warm up," let us observe that from the triangle inequality,
\[ \Delta = \frac{1}{N} Ed_H(V, Z) \leq \frac{1}{N} [Ed_H(V, Y) + Ed_H(Y, X) + Ed_H(X, Z)]. \]

Now
\[ Ed_H(Y, X) = D_1 N, \quad Ed_H(Z, X) = D_2 N. \]
\[ (16) \]

Furthermore,
\[ Ed_H(V, Y) = \sum_v E[d_H(v, Y) | V = v] Pr(V = v). \]

Now suppose that we are given \( V = v \in A_k \). Then, \( Y = g_i(k) \). Since \( Pr(V = v) = 2^{-N}, \)
\[ Ed_H(V, Y) = \sum_{k=1}^{M_1} \sum_{v \in A_k} 2^{-N} d_H[v, g_i(k)] \]
\[ = \sum_{k=1}^{M_1} Pr(X = x) d_H[x, g_i(k)] = ND_1. \]
\[ (17) \]

Thus,
\[ \Delta \leq 2D_1 + D_2. \]
\[ (18) \]

Substitution of (18) into (15a) yields \( (R_1, R_2, d_0, d_1, d_2) \)
\[ R_1 + R_2 \geq 2 - h(2d_1 + d_2) \]
which is the result reported in Ref. 1.

We will now establish a tighter bound on \( \Delta \).
\[ 2D_1 \leq 1, \]

so that (15) yields that for achievable \( (R_1, R_2) \),
\[ R_1 + R_2 \geq 2 - h\left( d_1 + 2d_1 - \frac{2d_1 - 1}{1 - h(\Delta)} \right) \]
which is (6b), the inequality required for Theorem 1.

**Upper bound on \( \Delta \):** We establish inequality (18) for \( 1 \leq k \leq M_1 \) be fixed. Let \( A_k \) be as defined in (15). Let \( \mu_k = \mu \). Let the members of \( A_k \) be the \( N \)-vector \( \mu \). Thus, when \( X = x \in A_k \), then \( Y = y \). Let \( A \), with \( m \)th row
\[ a_m = (a_{m1}, a_{m2}, \ldots, a_{mn}) = \]
where \( \oplus \) denotes modulo 2 vector addition. Since \( n \)th coordinates of \( x_m \) and \( y \) are different, and that
\[ \frac{1}{N} E[d_H(X, Y) | f_i(X) = k] \]
\[ = \frac{1}{N} \sum_{m=1}^{\mu} d_H(x_m, y) = \frac{1}{N} \mu \]
\[ = \frac{1}{N} \sum_{n=1}^{\mu} s_n, \]
where for \( 1 \leq n \leq N, \)
\[ s_n = \frac{1}{\mu} \sum_{m=1}^{\mu} a_{mn} \]
is the fraction of 1's in column \( n \) of \( A \).

Next, for \( 1 \leq m \leq \mu \), let \( z_m = g_i(x_m) \) results when \( X = x_m \). Let \( B \) be the \( \mu \times N \) array
\[ b_m = (b_{m1}, b_{m2}, \ldots, b_{mn}) = \]
we obtain
\[ \frac{1}{N} I(V; Z) + \frac{1}{N} I(X; X, Y, Z) \]
\[ \frac{1}{N} I(V; Z) + \frac{1}{N} I(X; X) \]
\[ 2 - h \left( \frac{Ed_H(V, Z)}{N} \right) - h \left( \frac{Ed_H(X, X)}{N} \right) \]
\[ 2 - h(\Delta) - h(d_0), \quad \text{(15a)} \]
\[ \Delta = \frac{Ed_H(V, Z)}{N}. \quad \text{(15b)} \]

The "rate-distortion bound" which states that if \( \mathbf{V} \) is uniformly distributed in \( \mathbb{B}^N \) (as are \( \mathbf{V} \) and \( \mathbf{X} \)), and \( Y \) is binary random vector, then \( I(U; \mathbf{U}) \geq 0 \). (See Ref. 4.)

An upper bound on \( \Delta \) in terms of \( D_1 \) and \( D_2 \).
\[ 1 / N \{ Ed_H(V, Y) + Ed_H(Y, X) + Ed_H(X, Z) \} \]
\[ = D_1 N, \quad Ed_H(Z, X) = D_2 N, \quad \text{(16)} \]
\[ \sum_v E[ d_H(v, Y) | V = v ] Pr(V = v). \]

For given \( \mathbf{V} = V \in A_k \). Then, \( \mathbf{Y} = g_i(k) \).
\[ = \sum_{v \in A_k} 2^{-N} d_H(v, g_i(k)) \]
\[ \sum_{x \in A_k} \Pr(X = x) d_H(x, g_i(k)) = ND_1, \quad \text{(17)} \]
\[ \Delta \leq 2D_1 + D_2. \quad \text{(18)} \]

Substitution of (18) into (15a) yields that for achievable \( (R_1, R_2, d_0, d_1, d_2) \)
\[ R_1 + R_2 \geq 2 - h(2d_1 + d_2) - h(d_0), \quad \text{(19)} \]
which is the result reported in Ref. 1.

We will now establish a tighter bound on \( \Delta \), namely, for \( D_2 + 2D_1 \leq 1, \)
\[ \Delta = \frac{1}{N} Ed_H(V, Z) \leq D_2 + 2D_1 - \frac{2D_1^2}{1 - D_2}, \quad \text{(20)} \]
so that (15) yields that for achievable \( (R_1, R_2, d_0, d_1, d_2) \)
\[ R_1 + R_2 \geq 2 - h \left( D_2 + 2d_1 - \frac{2d_1^2}{1 - D_2} \right) - h(d_0), \quad \text{(21)} \]
which is (6b), the inequality required for Theorem 2.

Upper bound on \( \Delta \): We establish inequality (20) as follows. Let \( k, 1 \leq k \leq M \), be fixed. Let \( A_k \) be as defined in (12), and let its cardinality \( \mu_k = \mu \). Let the members of \( A_k \) be the \( N \)-vectors \( \{x_m\}_{m=1}^\mu \). Let \( y = g_i(k) \).

Thus, when \( \mathbf{X} = \mathbf{x} \in A_k \), then \( \mathbf{Y} = \mathbf{y} \).
Now, form a \( \mu \times N \) array, \( \mathbf{A} \), with \( n \)th row
\[ \mathbf{a}_m = (a_{m1}, a_{m2}, \ldots, a_{mN}) = \mathbf{x}_m \oplus \mathbf{y}, \quad \text{(22)} \]
where \( \oplus \) denotes modulo 2 vector addition. Thus, \( a_{mn} = 1 \), when the \( n \)th coordinates of \( \mathbf{x}_m \) and \( \mathbf{y} \) are different, and \( a_{mn} = 0 \), otherwise. Note that
\[ \frac{1}{N} E[ d_H(X, Y) | f_i(X) = k ] \]
\[ = \frac{1}{N} \sum_{\mu} \sum_{m=1}^\mu d_H(x_m, y) = \frac{1}{N} \sum_{\mu} \sum_{m=1}^\mu a_{mn} \]
\[ = \frac{1}{N} \sum_{n=1}^N s_n, \quad \text{(23a)} \]
where for \( 1 \leq n \leq N, \)
\[ s_n = \frac{1}{\mu} \sum_{m=1}^\mu a_{mn} \quad \text{(23b)} \]
is the fraction of 1’s in column \( n \) of \( \mathbf{A} \).

Next, for \( 1 \leq m \leq \mu \), let \( \mathbf{z}_m = g_{z}f_i(x_m) \) be the value of \( \mathbf{Z} \) which results when \( \mathbf{X} = \mathbf{x}_m \). Let \( \mathbf{B} \) be the \( \mu \times N \) array with \( m \)th row
\[ \mathbf{b}_m = (b_{m1}, b_{m2}, \ldots, b_{mN}) = \mathbf{z}_m \oplus \mathbf{y}, \quad \text{(24)} \]
Then,
\[
\frac{1}{N} E[d_H(X, Z) | f_i(X) = k] = \frac{1}{N} \sum_{m=1}^{\mu} d_H(x_m, z_m) = \frac{1}{N} \mu \sum_m d_H(a_m, b_m)
\]
where the last equality follows from the fact that for \(a, b \in \{0, 1\}, a(1-b) + b(1-a) = 0 \) or 1 according as \(a = b\) or \(a \neq b\). Let \(\tau_n\) be the expression in braces in (25). Then,
\[
\tau_n = \frac{1}{\mu} \sum_{m=1}^{\mu} [a_{mn} + b_{mn} - 2a_{mn}b_{mn}]
\]
\[
\geq \frac{1}{\mu} \sum_{m} [a_{mn} - b_{mn}] = \frac{1}{\mu} \sum_{m} [b_{mn} - a_{mn} + 2a_{mn}(1-b_{mn})]
\]
\[
\geq \frac{1}{\mu} \sum_{m} (b_{mn} - a_{mn}) = \frac{1}{\mu} \sum_{m} (b_{mn} - a_{mn}) = t_n - s_n.
\]  
(26a)
where
\[
t_n = \frac{1}{\mu} \sum_{m=1}^{\mu} b_{mn}, \quad 1 \leq n \leq N,
\]  
(26b)
and \(s_n\) is given by (23b). We conclude that \(\tau_n \geq |t_n - s_n|\), so that (25) yields
\[
\frac{1}{N} E[d_H(X, Z) | f_i(X) = k] \geq \frac{1}{N} \sum_{n=1}^{N} |t_n - s_n|.
\]  
(27)
Finally, consider
\[
\frac{1}{N} E[d_H(V, Z) | f_i(X) = k]
\]
\[
= \frac{1}{N} \sum_{m=1}^{\mu} E[d_H(V, Z) | X = x_m]
\]
\[
= \frac{1}{N \mu} \sum_{m=1}^{\mu} E[d_H(V, z_m) | X = x_m]
\]
\[
= \frac{1}{N \mu} \sum_{m=1}^{\mu} \sum_v d_H(v, z_m) \Pr(V = v | X = x_m).
\]  
(28)
Now, from (13) which defines \(V\),
\[
\Pr(V = v | X = x_n) = \begin{cases} 
\mu^{-1}, & 0 < v < \mu, \\
0, & \text{otherwise}.
\end{cases}
\]
Thus, (28) yields,
\[
\frac{1}{N} E[d_H(V, Z) | f_i(X) = k]
\]
\[
= \frac{1}{N \mu^2} \sum_{m=1}^{\mu} \sum_{m'} d_H(x_m', z_m')
\]
\[
= \frac{1}{N \mu^2} \sum_{m,m'} d_H(a_m', b_m')
\]
\[
= \frac{1}{N \mu^2} \sum_{m,m'} \sum_n [a_{mn}(1 - b_{mn}) + b_{mn}(1 - a_{mn})] 
\]
\[
= \frac{1}{N} \sum_{n=1}^{N} [s_n(1 - t_n) + t_n(1 - s_n)].
\]
Now make the dependence on \(s_n\) and \(t_n\) explicit, and dropping \(t_n\) and \(t_{nk}\), respectively. Then, on averaging over all \(s_n\),
\[
D_1 = \frac{1}{N} E\{ d_H(X, Y) \} = \sum_{k=1}^{M_1} \Pr(f_i(X) = k) \times \frac{1}{N} = \sum_{k=1}^{M_1} \sum_{n=1}^{N} P(n, k)
\]
where \(P(n, k) = \Pr(f_i(X) = k) \times \frac{1}{N}\). Similarly
\[
D_2 = \frac{1}{N} E\{ d_H(X, Z) \} \geq \sum_{k=1}^{M_1} \sum_{n=1}^{N} P(n, k)\]
Finally, (29) becomes
\[
\Delta = \frac{1}{N} E\{ d_H(V, Z) \}
\]
\[
= \sum_{k=1}^{M_1} \sum_{n=1}^{N} P(n, k)[s_n(1 - t_{nk}) + t_n(1 - s_n)]
\]
We now apply the following inequality, the analogue of (19) in Section IV:
Let \(S, T\) be random variables such that \(0 \leq S \leq T \leq 1\), \(D_1, E\{ |T - S| \} \leq D_2\), with \(2D_1 + D_2 \leq 1. T\)
\[
E(S(1 - T) + T(1 - S)) \leq D_2 + D_1.
\]
Now, from (13) which defines V,
\[
\Pr(V = v | X = x_m) = \begin{cases} 
\mu^{-1}, & v \in A_m, \\
0, & \text{otherwise}.
\end{cases}
\]
Thus, (28) yields,
\[
\frac{1}{N} E[d_H(V, Z) | f_i(X) = k]
\]
\[
= \frac{1}{N\mu^2} \sum_{m=1}^{M_t} \sum_{m'=1}^{M_t} d_H(x_{m'}, x_m)
\]
\[
= \frac{1}{N\mu^2} \sum_{m=1}^{M_t} d_H(a_m, b_m)
\]
\[
= \frac{1}{N\mu^2} \sum_{m,m'} \sum_n [a_{m,n}(1 - b_{m,n}) + (1 - a_{m,n}) b_{m,n}]
\]
\[
= \frac{1}{N} \sum_{n=1}^{N} [s_n(1 - t_n) + t_n(1 - s_n)].
\]
(29)

Now make the dependence of s_n and t_n on k explicit by writing s_{nk} and t_{nk}, respectively. Then, on averaging over k, (23b) becomes

\[
D_1 = \frac{1}{N} E(d_H(X, Y)) = \sum_{k=1}^{M_t} \Pr(f_i(X) = k) \frac{1}{M_t} \sum_{n=1}^{N} s_{nk}
\]
\[
= \sum_{k=1}^{M_t} \sum_{n=1}^{N} P(n, k)s_{nk},
\]
(30a)

where \(P(n, k) = \Pr(f_i(X) = k) \times \frac{1}{N} \). Similarly, (27) becomes

\[
D_2 = \frac{1}{N} E(d_H(X, Z)) \geq \sum_{k=1}^{M_t} \sum_{n=1}^{N} P(n, k)|t_{nk} - s_{nk}|.
\]
(30b)

Finally, (29) becomes

\[
\Delta = \frac{1}{N} E(d_H(V, Z))
\]
\[
= \sum_{k=1}^{M_t} \sum_{n=1}^{N} P(n, k)[s_{nk}(1 - t_{nk}) + t_{nk}(1 - s_{nk})].
\]
(30c)

We now apply the following inequality, the proof of which is given in Section IV:

Let S, T be random variables such that 0 \leq S, T \leq 1, and E(S) \leq D_1, E(|T - S|) \leq D_2, with 2D_1 + D_2 \leq 1. Then,
\[
E(S(1 - T) + T(1 - S)) \leq D_2 + 2D_1 - \frac{2D_1^2}{1 - D_2}.
\]
(31)
Let $S, T$ be the random variables which take the value $s_{nk}, t_{nk}$, respectively, with probability $P(n, k)$, then (30) and (31) imply that, for $2D_1 + D_2 \leq 1$,
\[
\Delta \leq D_2 + 2D_1 - \frac{2D_1^2}{1 - D_2},
\]
which, when substituted in (15) gives (6b), proving Theorem 2.

**IV. PROOF OF THE INEQUALITY**

Define $Q(D_1, D_2)$ as the supremum of $E\{S(1 - T) + T(1 - S)\}$ over all distributions of $(S, T)$ on the unit square $[0, 1]^2$ for which $E\{S\} \leq D_1, E\{|T - S|\} \leq D_2$.

**Theorem 3:** (a) For $2D_1 + D_2 \leq 1$ one has
\[
Q(D_1, D_2) = 2D_1 + D_2 - \frac{2D_1^2}{1 - D_2},
\]
with $Q(0, 1) = 1$. (b) For $2D_1 + D_2 \geq 1$ one has
\[
Q(D_1, D_2) \geq \frac{1}{2}.
\]

To establish this, introduce for $S, T, x, y$ in $[0, 1], y \neq 1$, the function
\[
F(S, T, x, y) = S(1 - T) + T(1 - S) + \left(\frac{4x}{1 - y} - 2\right)(S - x) + \left[\frac{2x^2}{(1 - y)^2} - 1\right]|S - T| - y.
\]

**Lemma:** For $2x + y \leq 1, y < 1$ the maximum of $F(S, T, x, y)$ over all $(S, T)$ in $[0, 1]^2$ is $2x + y - 2x^2/(1 - y)$.

**Proof:** For fixed $S$, the maximum of $F$ over $T$ must, by piecewise linearity, be at either $T = 0, S$, or 1.

(i) If $T = 0$, then
\[
F = S + \left(\frac{4x}{1 - y} - 2\right)(S - x) + \left[\frac{2x^2}{(1 - y)^2} - 1\right](S - y),
\]
and this is maximized over $S$ at either $S = 0$ or $S = 1$.

(a) For $S = 0$
\[
F = 2x - \frac{4x^2}{1 - y} + y - \frac{2x^2y}{(1 - y)^2} = 2x + y - \frac{2x^2}{1 - y} - \frac{2x^2y}{(1 - y)^2} \leq 2x + y - \frac{2x^2}{1 - y}.
\]

(b) For $S = 1$
\[
F = 1 - 2 + 2x + \frac{4x(1 - x)}{1 - y} + \frac{2x^2}{1 - y} - 1 + \frac{2x^2}{1 - y} = 2x + y - \frac{2x^2}{1 - y} - 2 + \frac{4x}{1 - y} \leq 2x + y - \frac{2x^2}{1 - y}.
\]

(ii) If $T = S$, then
\[
F = 2S - 2S^2 + \left(\frac{4x}{1 - y} - 2\right)(S - x) + \frac{2x^2}{(1 - y)^2} - 1
\]
The maximum over $S$ of this quadratic is at $S = x \leq 1 - y$ and this holds as $x \leq 1 - y$.

maximum is
\[
\frac{2x}{1 - y} - \frac{2x^2}{(1 - y)^2} + \left(\frac{4x}{1 - y} - 2\right)\left(\frac{x}{1 - y} - xight) = 2x + y - \frac{2x^2}{1 - y}.
\]

(iii) If $T = 1$, then
\[
F = 1 - S + \left(\frac{4x}{1 - y} - 2\right)(S - x) + \frac{1}{(1 - y)^2} - 1
\]
which is maximized by taking $S$ either 0 or 1.

(a) For $S = 0$
\[
F = 2x + y - \frac{4x^2}{1 - y} + \frac{2x^2}{1 - y} = 2x + y - \frac{2x^2}{1 - y}
\]
(b) For $S = 1$
\[
F = 2x + y + \frac{4x}{1 - y} - \frac{4x^2}{1 - y} - 2 + \frac{4x}{1 - y} = 2x + y - \frac{2x^2}{1 - y} + \frac{4x}{1 - y} \leq 2x + y - \frac{2x^2}{1 - y},
\]

since $4x/(1 - y) \leq 2$.

Thus, the maximum is as stated and is obtained at $S = 0, T = y$ and for $T = 1, S = 0$. This completes the proof.

Turning to the proof of Theorem 3, consider $Q(D_1, D_2)$ on the unit square for which
random variables which take the value $s_{nk}, t_{nk}$, probability $P(n, k)$, then (30) and (31) imply that,

$$1 \leq D_2 + 2D_1 - \frac{2D_1^2}{1 - D_2}, \tag{32}$$

and in (15) gives (6b), proving Theorem 2.

**Duality**

the supremum of $E(S(1 - T) + T(1 - S))$ for all $(S, T)$ on the unit square $[0, 1]^2$ for which $D_2 \leq D_2.$

$D_2 = 2D_1 + D_2 - \frac{2D_1^2}{1 - D_2},$

$2D_1 + D_2 \geq 1$ one has

$Q(D_1, D_2) \geq \frac{1}{2}$.

Reduce for $S, T, x, y$ in $[0, 1], y \neq 1,$ the expression

$P = T(1 - S) + (S - x) \left[ \frac{2x^2}{(1 - y)^2} - 1 \right] |S - T| - y$. \quad (33)$

$y < 1$ the maximum of $F(S, T, x, y)$ over all $S \in [0, 1]$, $T \in [0, 1]$.

maximum of $F$ over $T$ must, by piecewise consideration, take place at either $S = 0$ or $S = 1,$

$$F = 1 - S + \left( \frac{4x}{1 - y} - 2 \right)(S - x) + \left[ \frac{2x^2}{(1 - y)^2} - 1 \right](1 - S - y),$$

which is maximized by taking $S$ either 0 or 1.

(a) For $S = 0,$

$$F = 2x + y - \frac{4x^2}{1 - y} + \frac{2x^2}{1 - y},$$

$$= 2x + y - \frac{2x^2}{1 - y}.$$ 

(b) For $S = 1,$

$$F = 2x + y + \frac{4x}{1 - y} - \frac{4x^2}{1 - y} - 2 - \frac{2x^2 y}{(1 - y)^2},$$

$$= 2x + y - \frac{2x^2}{1 - y} + \frac{4x}{1 - y} - 2 - \frac{2x^2 y}{(1 - y)^2},$$

$$\leq 2x + y - \frac{2x^2}{1 - y}.$$

Thus, the maximum is as stated and is attained for $T = S = x/(1 - y)$ and for $T = 1,$ $S = 0.$ This completes the proof of the lemma.

Turning to the proof of Theorem 3, consider any distribution of $(S, T)$ on the unit square for which

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If \( y = 1 \), then \( x = 0 \) from which it follows that \( S = 0, \ T = 1 \) almost surely, giving \( E\{S(1 - T) + T(1 - S)\} = 1 \).

If \( y < 1 \), one has, by the lemma,

\[
E\{F(S, T, x, y)\} = E\{S(1 - T) + T(1 - S)\} \leq 2x + y - \frac{2x^2}{1 - y}.
\]

If one chooses the distribution \( T = 1, \ S = 0 \) with probability \( y \) and \( T = S = x/(1 - y) \) with probability \( 1 - y \), equality is attained. This determines the maximum of \( E\{S(1 - T) + T(1 - S)\} \) subject to (34). As \( 2x + y - 2x^2/(1 - y) \) is monotone increasing (for \( 2x + y \leq 1 \)) in both \( x \) and \( y \), the maximum is unchanged if one allows all \( x, y \) with \( 0 \leq x \leq D_1, \ 0 \leq y \leq D_2 \). This establishes part (a) of Theorem 3.

For part (b), it suffices to observe that \( Q \) is monotone nondecreasing in both arguments by its definition and that on the boundary, where \( 2 D_1 + D_2 = 1 \), one has \( Q(D_1, D_2) = (1 + D_2)/2 \). This establishes part (b). (It could easily be shown that \( Q(D_1, D_2) = (1 + D_2)/2 \) for all \( (D_1, D_2) \) in the unit square satisfying \( 2 D_1 + D_2 = 1 \).

REFERENCES


Microprocessor Firmware Using Erasable Programmable Read-Only Memory (EPROM)

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Microprocessor-based systems are used in a number of modern telecommunications. The controlling processor systems is stored in firmware which allows reprogrammable read-only memory (ROM) or erasable programmable read-only memory (EPROM) chips to be used. As new services are implemented, the firmware is updated. Due to the cost of this process, it is desirable to have a centralized method of determining the firmware content. This paper presents a method of determining the resources required for a centralized inventory scheme.

I. INTRODUCTION

Microprocessor-based systems are used in a number of modern telecommunications. The controlling processor systems is stored in firmware which allows reprogrammable read-only memory (ROM), programmable read-only memory (PROM), or erasable programmable read-only memory (EPROM) chips to be used. As new services are implemented, the firmware is updated. Due to the cost of this process, it is desirable to have a centralized method of determining the firmware content. This paper presents a method of determining the resources required for a centralized inventory scheme.

One serious drawback to using EPROM firmware is that it is altered by means of a data link. A change in the firmware requires removal and reinstallation of the memory. Because of this manual process, updating the firmware in processor systems may involve a long and costly process.

The object of this paper is to provide