Selfish routing:
Networks, games and individual choice.

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Introduction
Network routing problems

An individual wishes to get from A to B in a network with minimum delay – which route do they take through the network?

Regime I

• System controller assigns a route to each individual.

• System attains system optimum.

Regime II

• Each individual chooses their route to minimize their own delay.

• System eventually settles to a user equilibrium.
Wardrop or user equilibrium

The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Wardrop, J.G. (1952)

Each user has an infinitesimal effect on the system.
Auckland, Monday 10 June, 8.30 a.m., actual traffic
Network with collection $R$ of $N$ routes from $A$ to $B$.

Probabilistic routing – user optimal/equilibrium policies

$p_r = \text{probability of taking route } r, \text{ with } p_r \geq 0, \sum_r p_r = 1.$

$p = (p_1, p_2, \ldots, p_N)$

$W_r(p) = \text{expected transit time via route } r \in R.$

At a user equilibrium, $p^{EQ}$, there exists $c$ such that

$W_r(p^{EQ}) = c \quad \text{if } p_r^{EQ} > 0$

$\geq c \quad \text{if } p_r^{EQ} = 0.$
State dependent routing – user optimal/equilibrium policies

A decision policy $D$ is a partition of state space, $S$, into sets $D_r$, $r \in R$ such that if system is in state $n \in D_r$ when a user arrives, then they take route $r$.

For a policy $D \in \mathcal{D}$ and $n \in S$, $z^D_r(n)$ is expected time to reach the destination for a general user, if system is in state $n$ immediately prior to their arrival, and they choose to take route $r$.

A policy $D \in \mathcal{D}$ is a user optimal policy or user equilibrium if for each $n \in S$

$$n \in D_r \implies z^D_r(n) \leq z^D_s(n) \text{ for all } s \neq r, s \in R.$$
Downs-Thomson network
Downs-Thomson network

$Q_1$: 1 server, $\mu_1$

$Q_2$: $\infty$ server, $\mu_2$

Two Poisson arrival streams
- dedicated users to queue 2 at rate $\lambda_2$,
- general users at rate $\lambda$.

General users choose route
- either probabilistic or state-dependent routing.

$Q_1$ single server queue ($\cdot|M|1$), exponential service times, mean $1/\mu_1$.
$Q_2$ batch service $\infty$ server queue, service times with mean $1/\mu_2$. 
• Single server queue – private transportation (e.g. cars).
  – delay increases as load increases

• Batch service queue – public transportation (e.g. shuttle bus).
  – delay decreases as load increases
  – frequency of service increases as load increases

• This version of model first proposed by Calvert (1997) as queueing network version of transportation model that gives rise to the Downs Thomson paradox.

• Paradox is that delays for all users can increase when capacity of private transportation (roading) is increased. First observed by Downs (1962) and Thomson (1977).

• Afimeimounaga, Solomon, Z (2005, 2010)
Downs-Thomson network –
probabilistic routing
\[ \lambda \xrightarrow{p} Q_1 \xrightarrow{1-p} Q_2 \]

\[ Q_1: 1 \text{ server, } \mu_1 \]

\[ Q_2: \infty \text{ server, } \mu_2 \]

\[ Q_1 \text{ single server queue (} \cdot | M | 1 \text{). Expected delay } W_1 = \frac{1}{\mu_1 - \lambda p} \]

\[ Q_2 \text{ batch service } \infty \text{ server queue. Expected delay } W_2 = \frac{1}{\mu_2} + \frac{N-1}{2(\lambda_2 + \lambda(1-p))} \]

Both \( W_1 \) and \( W_2 \) are increasing in \( p \).
\( \mu_1 = 0.8 \)
\( \lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3 \)
\( W_1, - - - - - - - - , W_2, \)
\( \mu_1 = 0.8 \)

\( \lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3 \)

\( W_1, \ldots, W_2, \quad \)
\[ \mu_1 = 0.8 \]
\[ \lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3 \]
\[ W_1, \quad W_2, \]
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\[ W_1, \quad , W_2, \]
\[ \mu_1 = 0.8 \]

\[ \lambda = 1, \lambda_2 = 0.1, \mu_2 = 1, N = 3 \]

\[ W_1, \ldots, \ldots, W_2, \]
\mu_1 = 0.8
\lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3
W_1, - - - - - - - - , W_2, -----
\[ \mu_1 = 0.8, 0.95 \]
\[ \lambda = 1, \lambda_2 = 0.1, \mu_2 = 1, N = 3 \]
\[ W_1, \quad \cdots, \quad W_2, \]
\[ \mu_1 = 0.8, 0.95 \]
\[ \lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3 \]
\[ W_1, \quad W_2, \quad \]
\[
\mu_1 = 0.8, 0.95, 1.05 \\
\lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3 \\
W_1, \quad W_2
\]
\[ \mu_1 = 0.8, 0.95, 1.05 \]
\[ \lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3 \]
\[ W_1, - - - - - - - - , W_2, \]
$\lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3$

$W = p^{EQ}W_1 + (1 - p^{EQ})W_2$
\[ \lambda = 1, \lambda_2 = 0.1, \mu_2 = 1, \, N = 3 \]

\[ W = p^{EQ}W_1 + (1 - p^{EQ})W_2 \]

\[ W = \min_p pW_1 + (1 - p)W_2 \]
Multiple equilibria

\[ \lambda = 1, \lambda_T = 0.1, \mu_T = 3, N = 2 \]

\( W(p^*_R) \), delay at user equilibrium

\(-- -- --\) delay at socially optimal equilibrium.
Summary

- Delay at user equilibria under probabilistic routing can be arbitrarily large ($\rightarrow \infty$ as $\lambda_2 \rightarrow 0$).

- Probabilistic routing can be viewed as users having limited information about the system – mean delays at queues only.

- There may be multiple equilibria (for $N = 2$ when queue 1 is $|M|1$, more generally when queue 1 is $|G|1$).

- What happens if users have information about instantaneous state of network?

- Challenge in answering this question is that expected delay depends not just on current state of network, but also on decisions made by future arrivals (similar to networks with overtaking).

Downs-Thomson network –
state dependent routing
State dependent policies

\[ X_1(t) = \text{number of customers in queue 1} \]
\[ \text{(including customer in service)} \]

\[ X_2(t) = \text{number of customers waiting for service in queue 2} \]
\[ \text{(not including those in service)} \]

State space \( S = \mathbb{Z}_+ \times \{0, 1, 2, \ldots, N - 1\} \).

Process \( X_D \) operating under decision policy \( D \) has transition rates:

\[
\begin{align*}
    n &\rightarrow \left\{ \begin{array}{ll}
    n - e_1 & \text{at rate } \mu_1 I\{n_1 > 0\} \\
    n + e_1 & \text{at rate } \lambda I\{n \in D_1\} \\
    (n_1, (n_2 + 1) \mod N) & \text{at rate } \lambda_2 + \lambda I\{n \in D_2\}
    \end{array} \right.
\end{align*}
\]

where \( I_A = 1 \) if \( A \) occurs, and \( I_A = 0 \) otherwise.

A policy \( D \in \mathcal{D} \) is a user optimal policy or user equilibrium if

\[ n \in D_1 \iff z_1^D(n) < z_2^D(n) \quad \text{for all } n \in S. \]
Points in $D_1$ are indicated by •. Points in $D_2$ are indicated by o.

Unique user optimal policy for

$N = 10, \lambda = 1.5, \lambda_2 = 0.5, \mu_1 = 2, \mu_2 = 1$.

A policy $D \in \mathcal{D}$ is monotonic if $D$ satisfies

(A) $\mathbf{n} \in D_2 \Rightarrow \mathbf{n} + e_1 \in D_2$ for all $\mathbf{n} \in S$ and

(B) $\mathbf{n} \in D_2 \Rightarrow \mathbf{n} + e_2 \in D_2$ for all $\mathbf{n} \in S$
Main results

- A user optimal policy exists and is unique (no randomization needed).
- The user optimal policy is monotonic.
- The user optimal policy is monotonic in the parameters $\lambda, \lambda_2, \mu_1, \mu_2$ in the following sense. Let $X^{(1)}$ and $X^{(2)}$ be two processes, with common batch size $N$ and user optimal policies $D^*(1), D^*(2)$ respectively. If $\lambda^{(1)} \geq \lambda^{(2)}, \mu_1^{(1)} \leq \mu_1^{(2)}, \lambda_2^{(1)} \geq \lambda_2^{(2)}$ and $\mu_2^{(1)} \geq \mu_2^{(2)}$, then $D^*_1(1) \subset D^*_1(2)$.
- Proof uses a coupling argument.
- As part of the proof show monotonicity of $z_2^D(n)$ in $\lambda, \lambda_2, \mu_1, \mu_2$; and in the decision policy.
$\lambda = 1, \lambda_2 = .1, \mu_2 = 1, N = 3$

$W = p^{EQ}W_1 + (1 - p^{EQ})W_2$

$W = \min_p pW_1 + (1 - p)W_2$
Expected transit times under user optimal policy for state-dependent routing (———–), and probabilistic routing (−−−−−−−)

\[ \lambda = 1, \lambda_2 = 0.1, \mu_2 = 1, N = 3 \] for \( 0 \leq \mu_1 \leq 3 \).
Expected transit times under user optimal policy for state-dependent routing (———–), and probabilistic routing (−−−−−−−)

\( \lambda = 1, \lambda_2 = 0.1, \mu_2 = 1, N = 3 \) for \( 0 \leq \mu_1 \leq 3 \).
Summary

- For this network possible to show user equilibrium exists, is unique and monotonic.

- Numerical results indicate state dependent routing can mitigate worst effects of probabilistic routing. That is, more information leads to improved performance.

- Improvement in performance may be of practical interest – webcams, GPS navigation ....

- Does it hold for other networks?

- Afimeimounga, Solomon, Z (2010)
Variations
Two batch-service queues

Expected transit times under user optimal policy for state-dependent routing (———), and probabilistic routing (−−−−−−−) for 0 ≤ μ₁ ≤ 6.

λ = 4, λ₁ = 3, λ₂ = 1, μ₂ = 2, N₁ = N₂ = 5

Chen, Holmes, Z(2012)
Processor sharing queues

- Iterative procedure may converge to periodic orbit
- User equilibrium doesn’t always possess monotonicity properties
- Randomization needed
Braess’s paradox – original network

A: 1 server, \( \mu_1 \)

B: \( \infty \) server, \( \mu_2 \)

C: \( \infty \) server, \( \mu_2 \)

D: 1 server, \( \mu_1 \)
Braess’s network – augmented with additional route

Take

- Route 1 $A \rightarrow B$ with probability $p_1$.
- Route 2 $C \rightarrow D$ with probability $p_2$.
- Route 3 $A \rightarrow D$ with probability $p_3$. 
The arrival rate, \( \lambda \), varies from 0.10 to 4.75 (\( \lambda = 5.00 \) is upper bound on capacity of the system). Cohen and Kelly (1990)

Expected transit time: \( \mu_1 = \mu_4 = 2.5, \mu_2 = \mu_3 = 0.5 \)
Expected transit time with state-dependent routing:
\[ \mu_1 = \mu_4 = 2.5, \mu_2 = \mu_3 = 0.5 \]
(Calvert, Solomon, Z (1997))
Some final comments

- Do user equilibria exist more generally under state dependent routing, and if yes, when are they unique?

- How to overcome poor performance at user equilibria?

- Does more information lead to shorter delays in general? Effects of partial information

- Add monetary and other costs to the problem, as well as delays

- Convergence issues – effect of delayed information.

- Differing information and/or policies for different customer classes


