Hybrid Coding: A New Paradigm for Relay Communication

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***Abstract*—Motivated by recent advances in joint source–channel coding over networks, hybrid (analog/digital) coding has been proposed as a coding technique for discrete memoryless relay networks, whereby each relay transmits a symbol-by-symbol function of the received sequence at the channel output and its quantized version (analog-to-analog/digital interface). For a few simple channel models, it has been demonstrated that hybrid coding unifies both amplify–forward and noisy network coding and can strictly outperform both. This paper extends these results to the class of general *layered* relay networks and establishes the hybrid coding lower bound on the capacity for the class.**

***Index Terms*—Hybrid coding, Network Information Theory, Relay Networks.**

I. Introduction

Relaying is a fundamental building block in multihop cooperative communication systems. Over the past decades, three dominant paradigms have been proposed for relay communication: decode–forward, compress–forward, and amplify–forward.

- In decode–forward, each relay in the network recovers the transmitted message by the source and forwards it to the receiver (digital-to-digital interface) while coherently cooperating with the source node. Decode–forward was originally proposed in [1] for the relay channel and has been generalized to multiple relay networks, for example, in [2], [3] and further improved by combining it with structured coding [4], [5].
- In amplify–forward, each relay sends a scaled version of its received sequence and forwards it to the receiver (analog-to-analog interface). Amplify–forward was proposed in [6] for the Gaussian two–relay diamond network and subsequently studied for the Gaussian relay channel in [7]. Generalizations of amply–forward to general nonlinear analog mappings for relay communication have been proposed, for example, in [8].
- In compress–forward, each relay vector-quantizes its received sequence and forwards it to the receiver (analog-to-digital interface). Compress–forward was proposed in [1] for the relay channel and has been generalized to arbitrary noisy networks in [9] as noisy network coding; see also [10].

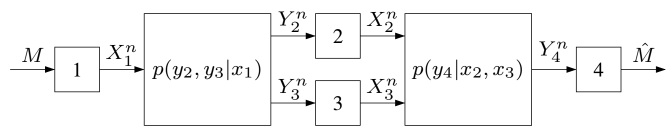

Fig. 1. The two-relay diamond network — an example of a 4-node layered network where the set of nodes $\{1, 2, 3, 4\}$ is partitioned into three subsets $\mathcal{L}_1 = \{1\}$, $\mathcal{L}_2 = \{2, 3\}$, and $\mathcal{L}_3 = \{4\}$.

In [11] we presented a new coding scheme for relay networks based on the hybrid coding architecture which was introduced in [12] in the context of lossy communications over multiple access channels. In this hybrid coding architecture, each relay node in the network transmits a symbol-by-symbol function of the received sequence and its quantized version (analog-to-analog/digital interface). We proved via two specific examples, the two–way relay channel and the two–relay diamond network, that hybrid coding can strictly outperform the existing coding schemes, not only amplify–forward and compress–forward, but also decode–forward.

In this paper, we generalize the existing results on hybrid coding by providing a general capacity lower bound for the class of *layered* relay networks, i.e., the networks where the nodes can be grouped into subsets $\mathcal{L}_1, \ldots, \mathcal{L}_L$ such that the source and destination nodes belong to $\mathcal{L}_1$ and $\mathcal{L}_L$, respectively, and there are links only between nodes in adjacent layers $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$, $i = 1, \ldots, L-1$. The two-relay diamond network depicted in Fig. 1 is an example of 3-layer networks.

The rest of the paper is organized as follows. In Section II we develop the necessary background on hybrid coding by revisiting a a capacity lower bound for the two-relay diamond channel presented in [11]. Section III presents the main result of the paper, a coding theorem for a general noisy layered relay network. Throughout the paper, we use the notation in [13].

II. Foundations of Hybrid Coding: The Diamond Relay Network

A canonical channel model used to feature the benefits of node cooperation in relay networks is the diamond channel introduced in [6] and depicted in Fig. 1. This two-hop network consists of a source node (node 1) that wishes to send a message $M \in [1:2^{nR}]$ to a destination (node 4) with the help of two relay nodes (nodes 2 and 3). The source node is connected through the broadcast channel $p(y_2, y_3|x_1)$ to the two relay nodes that are in turn connected to the destination node through the multiple-access channel $p(y_4|x_2, x_3)$. The capacity of the diamond channel is not known in general. Schein and Gallager [6] characterized inner bounds on the capacity region based on decode–forward, amplify–forward, and compress–forward. In this section, we introduce hybrid coding by revisiting a lower bound on the capacity region of the diamond relay network presented in [11].

A. Problem Statement and the Hybrid Coding Lower Bound

A diamond channel $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3, p(y_2, y_3|x_1) p(y_4|x_2, x_3), \mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}_4)$ consists of six alphabet sets and a collection of conditional pmfs on $\mathcal{Y}_2 \times \mathcal{Y}_3 \times \mathcal{Y}_4$. A $(2^{nR}, n)$ code for the diamond channel consists of

- a message set $[1:2^{nR}]$,
- an encoder that assigns a codeword $x_1^n(m)$ to each message $m \in [1:2^{nR}]$,
- two relay encoders, where relay encoder $j = 2, 3$ assigns a symbol $x_{j,i}(y_j^{i-1})$ to each past received output sequence $y_j^{i-1} \in \mathcal{Y}_j^{i-1}$, and
- a decoder that assigns an estimate $\hat{m}$ or an error message to each received sequence $y_4^n \in \mathcal{Y}_4^n$.

We assume that the message M is uniformly distributed over $[1:2^{nR}]$. The average probability of error is defined as $P_e^{(n)} = \mathsf{P}\{\hat{M} \neq M\}$. A rate R is said to be achievable for the diamond channel if there exists a sequence of $(2^{nR}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. The capacity C of the diamond channel is the supremum of the achievable rates R.

Hybrid coding yields the following lower bound on the capacity, the proof of which can be found in [14].

Theorem 1 ([11]): The capacity of the diamond channel $p(y_2, y_3|x_1)p(y_4|x_2, x_3)$ is lower bounded as

$$\begin{aligned} C \geq \max\min\{&I(X_1; U_2, U_3, Y_4), \\ &I(X_1, U_2; U_3, Y_4) - I(U_2; Y_2|X_1), \\ &I(X_1, U_3; U_2, Y_4) - I(U_3; Y_3|X_1), \\ &I(X_1, U_2, U_3; Y_4) - I(U_2, U_3; Y_2, Y_3|X_1)\}, \end{aligned} \tag{1}$$

where the maximum is over all conditional pmfs $p(x_1)p(u_2|y_2)p(u_3|y_3)$ and functions $x_2(u_2, y_2)$, $x_3(u_3, y_3)$.

B. Hybrid Coding Architecture

The proof of achievability of Theorem 1 is based on a hybrid coding architecture depicted in Fig. 2, which is used at relay node $j = 2, 3$. At the source node, the message M is mapped to one of 2^{nR_1} sequences $X_1^n(M)$ i.i.d. $\sim p(x_1)$ as in point-to-point communication. At the relay nodes, the "source" the sequence Y_j^n, $j = 2, 3$, is separately mapped into one of 2^{nR_j} independently generated sequences $U_j^n(M_j)$. Then, the pair

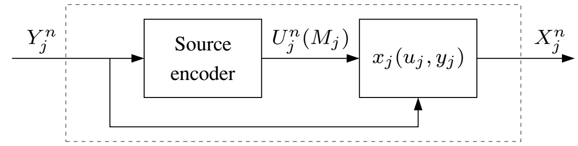

Fig. 2. Hybrid coding interface for relays.

$(Y_j^n, U_j^n(M_j))$ is mapped by node j to X_j^n via a symbol-by-symbol map. By the covering lemma, the source encoding operation at the relays is successful if

$$\begin{aligned} R_2 &> I(U_2; Y_2) \\ R_3 &> I(U_3; Y_3). \end{aligned}$$

At the destination node, decoding is performed by joint typicality and indirect decoding of the sequences (U_2^n, U_3^n), that is, by searching for the unique message $\hat{M} \in [1:2^{nR}]$ such that the tuple $(X_1^n(\hat{M}), U_2^n(M_2), U_3^n(M_3), Y_4^n)$ is typical for some $M_2 \in [1:2^{nR_2}]$ and $M_3 \in [1:2^{nR_3}]$. By the packing lemma, combined with the technique introduced in [12], the channel decoding operation at the destination node is successful if

$$\begin{aligned} R &< I(X_1; U_2, U_3, Y_4) \\ R + R_2 &< I(X_1, U_2; U_3, Y_4) + I(X_1; U_2) \\ R + R_3 &< I(X_1, U_3; U_2, Y_4) + I(X_1; U_3) \\ R + R_2 + R_3 &< I(X_1, U_2, U_3; Y_4) + I(X_1; U_2) \\ &\quad + I(X_1, U_2; U_3). \end{aligned}$$

Hence, the lower bound (1) is obtained by combining the conditions for source coding at the relay nodes with those for channel decoding at the destination and by eliminating the auxiliary rates (R_1, R_2) from the resulting system of inequalities.

C. Comparison with Other Relaying Strategies

Next, we show that hybrid coding can strictly outperform noisy network coding (and compress-forward). To do so, we further specialize to the case of a deterministic diamond channel, wherein the multiple access channel $p(y_2, y_3|x_1)$ and the broadcast channel $p(y_4|x_2, x_3)$ depicted in Fig. 1 are deterministic. Then, Theorem 1 yields the following inner bound.

Corollary 1: The capacity of the deterministic diamond channel is lower bounded as

$$C \geq \max_{p(x_1)p(x_2|y_2)p(x_3|y_3)} R(Y_2, Y_3, Y_4|X_2, X_3), \tag{2}$$

where

$$\begin{aligned} R(Y_2, Y_3, Y_4|X_2, X_3) = \min\{&H(Y_2, Y_3), H(Y_4), \\ &H(Y_2) + H(Y_4|X_2, Y_2), \\ &H(Y_3) + H(Y_4|X_3, Y_3)\}. \end{aligned}$$

Proof: Set in Theorem 1 $U_2 = (Y_2, X_2)$, $U_3 = (Y_3, X_3)$, $x_2(u_2, y_2) = x_2$, and $x_3(u_3, y_3) = x_3$ under a pmf $p(x_2|y_2)p(x_3|y_3)$. ∎

We can compare the result in Corollary 1 with the existing inner and outer bounds for this channel model. An outer bound on the capacity region is given by the cutset bound [15], which in this case simplifies to

$$C \leq \max_{p(x_1)p(x_2,x_3)} R(Y_2, Y_3, Y_4|X_2, X_3) \tag{3}$$

On the other hand, specializing the scheme in [16] for deterministic relay networks, we obtain the lower bound

$$C \geq \max_{p(x_1)p(x_2)p(x_3)} R(Y_2, Y_3, Y_4|X_2, X_3). \tag{4}$$

Note that (2), (3), and (4) differ only in the set of allowed maximizing input pmfs. In particular, (2) improves upon the inner bound (4) by allowing X_2 and X_3 to depend on Y_2 and Y_3 and thereby increasing the set of distributions $p(x_2, x_3)$. The following example demonstrates that the inclusion can be strict.

Example 1: Suppose that $p(y_2, y_3|x_1)$ is the Blackwell broadcast channel (i.e., $X_1 \in \{0, 1, 2\}$ and $p_{Y_2,Y_3|X_1}(0, 0|0) = p_{Y_2,Y_3|X_1}(0, 1|1) = p_{Y_2,Y_3|X_1}(1|2) = 1$) and $p(y_4|x_2, x_3)$ is the binary erasure multiple access channel (i.e., $X_2, X_3 \in \{0, 1\}$ and $Y_4 = X_2 + X_3 \in \{0, 1, 2\}$). It can be easily seen that the general lower bound reduces to $C \geq 1.5$, while the capacity is $C = \log 3$, which coincides with the hybrid coding lower bound (with $X_2 = Y_2$ and $X_3 = Y_3$). Thus, hybrid coding strictly outperforms the coding scheme by Avestimehr, Diggavi, and Tse [16] and noisy network coding [9].

III. Hybrid Coding in Layered Relay Networks

In this section, we present our main result, namely, the extension of Theorem 1 to general *layered* relay networks, i.e., networks with a single source node (say, node 1), a single destination (say, node N) and a channel pmf of the form

$$p(y_2, \ldots, y_N|x_1, \ldots, x_{N-1}) = \prod_{i=2}^{L} p(y(\mathcal{L}_i)|x(\mathcal{L}_{i-1})), \tag{5}$$

where $\mathcal{L}_1, \ldots, \mathcal{L}_L$ are a partition of the set $[1 : N]$ such that $\mathcal{L}_1 = \{1\}$ and $\mathcal{L}_L = \{N\}$. Throughout this section we use the notation $x(\mathcal{L}) = \{x_j, j \in \mathcal{L}\}$. As an example, the diamond network studied in Section II is a layered network with $L = 3$ layers, because the set of nodes $[1 : 4]$ is partitioned into a source node $\mathcal{L}_1 = \{1\}$, one set of relay nodes $\mathcal{L}_2 = \{2, 3\}$, and one destination node $\mathcal{L}_3 = \{4\}$, and the corresponding channel pmf factorizes as $p(y_2, y_3|x_1)p(y_4|x_2, x_3)$. In a layered relay network with L layers, each relay node $j \in \mathcal{L}_i$, $i \in [1 : L-1]$, uses the hybrid coding interface depicted in Fig. 2 to transmit the channel output Y_j to the nodes in the $(i+1)$-st layer. The main result of the paper is the following capacity lower bound for general layered relay networks based on hybrid coding.

Theorem 2: The capacity of the L-layered DMN (5) is lower bounded as

$$C \geq \max \min_{\mathcal{S} \subseteq [2:N-1]} I(X_1, U(\mathcal{S}); U(\mathcal{S}^c), Y_N) - \sum_{k \in \mathcal{S}} I(U_k; Y_k|U(\mathcal{S}_k), X_1), \tag{6}$$

where $\mathcal{S}_k = \{k' \in \mathcal{S} : k' < k\}$ and the maximum is over all conditional pmfs $p(x_1)\prod_{k=2}^{N-1} p(u_k|y_k)$ and functions $x_k(u_k, y_k)$, $k = 2, \ldots, N-1$.

Notice that in the case of the diamond network there are four subsets of the set of relay nodes $\{2, 3\}$. Accordingly, when specialized to this channel model the right hand side of (6) simplifies and yields the four expressions at the right hand side of (1).

A. Comparison with Other Relaying Strategies

Theorem 1 includes both the noisy network coding inner bound, which is recovered by setting $U_j = (X_j, \hat{Y}_j)$ with $p(x_j)p(\hat{y}_j|y_j)$, $j = 2, 3$, and the amplify–forward inner bound, which is obtained by setting $U_j = \emptyset$ for $j = 2, 3$. To visualize the differences between hybrid coding and noisy network coding, it is worth focusing on the special case of a deterministic layered networks, whereby $p(y(\mathcal{L}_i)|x(\mathcal{L}_{i-1}))$, $i \in [2 : L]$ are deterministic, i.e., the channel outputs are functions of the corresponding inputs. In this case, Theorem 2 simplifies to the following corollary.

Corollary 2: The capacity of the L-layered deterministic network is lower bounded as

$$C \geq \max \min_{\mathcal{S} \subseteq [2:N-1]} R(\mathcal{S}),$$

where

$$R(\mathcal{S}) = H(Y(\mathcal{S}^c), Y_N|X(\mathcal{S}^c)) + I(X_1, X(\mathcal{S}), Y(\mathcal{S}); X(\mathcal{S}^c)) - \sum_{\mathcal{S}} H(Y_k|X(\mathcal{S}_k), Y(\mathcal{S}_k), X_1)$$

and the maximum is over all conditional pmfs $p(x_1)\prod_{k=2}^{N-1} p(x_k|y_k)$, $k = 2, \ldots, N-1$.

Proof: Let $U_k = (X_k, Y_k)$, $k = 2, \ldots, N-1$ where $X_k \sim p(x_k|y_k)$. Then,

$$\begin{aligned}
&I(X_1, U(\mathcal{S}); U(\mathcal{S}^c), Y_N) - \sum_{\mathcal{S}} I(U_k; Y_k|U(\mathcal{S}_k), X_1) \\
&= I(X_1, X(\mathcal{S}), Y(\mathcal{S}); X(\mathcal{S}^c), Y(\mathcal{S}^c), Y_N) \\
&\quad - \sum_{\mathcal{S}} I(X_k, Y_k; Y_k|X(\mathcal{S}_k), Y(\mathcal{S}_k), X_1) \\
&= I(X_1, X(\mathcal{S}), Y(\mathcal{S}); Y(\mathcal{S}^c), Y_N|X(\mathcal{S}^c)) \\
&\quad + I(X_1, X(\mathcal{S}), Y(\mathcal{S}); X(\mathcal{S}^c)) \\
&\quad - \sum_{\mathcal{S}} H(Y_k|X(\mathcal{S}_k), Y(\mathcal{S}_k), X_1) \\
&= I(X_1, X(\mathcal{S}); Y(\mathcal{S}^c), Y_N|X(\mathcal{S}^c)) \\
&\quad + I(X_1, X(\mathcal{S}), Y(\mathcal{S}); X(\mathcal{S}^c)) \\
&\quad - \sum_{\mathcal{S}} H(Y_k|X(\mathcal{S}_k), Y(\mathcal{S}_k), X_1) \\
&= H(Y(\mathcal{S}^c), Y_N|X(\mathcal{S}^c)) + I(X_1, X(\mathcal{S}), Y(\mathcal{S}); X(\mathcal{S}^c)) \\
&\quad - \sum_{\mathcal{S}} H(Y_k|X(\mathcal{S}_k), Y(\mathcal{S}_k), X_1)
\end{aligned}$$

■

Note that the noisy network coding lower bound for deterministic networks is given by

$$C \geq \max \min_{\mathcal{S} \subseteq [2:N-1]} H(Y(\mathcal{S}^c), Y_N | X(\mathcal{S}^c))$$

where the maximization is over all $\prod_{k=1}^{N-1} p(x_k)$. Compared to the noisy network coding lower bound, hybrid coding has two additional terms while the maximization is done over the more general set of distributions $p(x_1) \prod_{k=2}^{N-1} p(x_k|y_k)$, $k = 2, \ldots, N-1$. The inclusion is in general strict, as demonstrated by Example 1. It can verified, in fact, that Corollary 2 reduces to Corollary 1 in the special case of the deterministic diamond channel.

IV. Concluding Remarks

Hybrid coding is a general coding technique for discrete memoryless relay networks, whereby each relay transmits a symbol-by-symbol function of the received sequence at the channel output and its quantized version (analog-to-analog/digital interface). In this paper, we first revisited the foundations of hybrid coding for relay networks by focusing on the two-relay diamond channel, for which hybrid coding (analog-to-analog/digital interface) yields a capacity lower bound that strictly outperform both amplify–forward (analog-to-analog interface) and compress–forward/noisy network coding (analog-to-digital interfaces). We then presented a capacity lower bound based on hybrid coding for general layered relay networks that naturally extends the previous result on the diamond channel.

While we assumed that the relay nodes do not attempt to decode the message transmitted by the source, the presented results can be further improved by combining hybrid coding with other coding techniques such as decode–forward and structured coding [4]. In this case, hybrid coding provides a general analog/digital-to-analog/digital interface for relay communication. In principle, hybrid coding can also be applied to the relay channel and other nonlayered relay networks. In this case, however, hybrid coding (or even amplify–forward) would not yield inner bounds on the capacity region in a single-letter form, due to the dependency between the channel input at each relay node and the previously received analog channel outputs.

Acknowledgments

This work was supported in part by the National Science Foundation Grant CCF-1117728.

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