On the Log Determinant of Non-Central Wishart Matrices

Young-Han Kim[1]
Information Systems Laboratory
Stanford University
Stanford, CA 94305, USA
e-mail: yhk@stanford.edu

Amos Lapidoth
Signal and Information Processing Lab.
Swiss Federal Inst. of Technology (ETH)
CH-8092 Zurich, Switzerland
e-mail: lapidoth@isi.ee.ethz.ch

Abstract — **We show that the expected log determinant of a complex non-central Wishart matrix is an increasing function of the non-centrality parameter. This demonstrates that the mutual information corresponding to an isotropically distributed Gaussian input to a multi-antenna Ricean fading channel is non-decreasing in the line-of-sight component.**

We consider a memoryless Ricean fading channel with n transmitter antennas and m receiver antennas whose time-k output $\mathbf{Y}_k \in \mathcal{C}^m$ is given by

$$\mathbf{Y}_k = \mathbb{H}_k \mathbf{x}_k + \mathbf{Z}_k, \quad k = 1, 2, \ldots, \tag{1}$$

where the vector $\mathbf{x}_k \in \mathcal{C}^n$ denotes the channel input at time k, the additive noise process $\{\mathbf{Z}_k\}$ is i.i.d. $\sim \mathcal{N}_\mathcal{C}(0, \mathcal{N}\,\mathsf{I}_m)$, i.e., temporally and spatially white zero-mean variance-$\mathcal{N}$ circularly symmetric complex Gaussian, and the channel matrix process $\{\mathbb{H}_k\} = \{\tilde{\mathbb{H}}_k + \mathsf{D}\}$ is independent of $\{\mathbf{Z}_k\}$ and temporally and spatially white unit-variance circularly symmetric complex Gaussian with the fading part $\tilde{\mathbb{H}}_k$ i.i.d. $\sim \mathcal{N}_\mathcal{C}(0, \mathsf{I}_m \otimes \mathsf{I}_n)$ and the deterministic line-of-sight gain part $\mathsf{D} \in \mathcal{C}^{m\times n}$. We assume that the receiver has perfect knowledge of the realization of $\{\mathbb{H}_k\}$. Under an isotropically distributed Gaussian input $\mathbf{X} \sim \mathcal{N}_\mathcal{C}(\mathbf{0}, \mathcal{E}_s\,\mathsf{I}_n)$, the mutual information between the channel terminals is given by

$$I(\mathbf{X}; \mathbf{Y}, \mathbb{H}) = \mathsf{E}\left[\log\det\left(\mathsf{I}_m + \frac{\mathcal{E}_s}{\mathcal{N}}\mathbb{H}\mathbb{H}^\dagger\right)\right]. \tag{2}$$

Since $\det\left(\mathsf{I}_m + \frac{\mathcal{E}_s}{\mathcal{N}}\mathbb{H}\mathbb{H}^\dagger\right) = \det\left(\mathsf{I}_n + \frac{\mathcal{E}_s}{\mathcal{N}}\mathbb{H}^\dagger\mathbb{H}\right)$, we assume without loss of generality that $m \le n$, i.e., the number of receiver antennas does not exceed the number of transmitter antennas. Under this condition, the random matrix $\mathbb{W} \triangleq \mathbb{H}\mathbb{H}^\dagger$ has the complex non-central Wishart distribution $\mathcal{W}_n(m, \Lambda)$ [1], which is the matrix analogue of the non-central χ^2 distribution. In addition, since the distribution of the random matrix $\tilde{\mathbb{H}}$ is rotationally symmetric, the mutual information (2) is a symmetric function of eigenvalues of $\Lambda \triangleq \mathsf{D}\mathsf{D}^\dagger$, and hence we assume that Λ is a nonnegative diagonal matrix with ordered diagonal entries $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m$.

While the isotropically distributed input $\mathbf{X}$ need not achieve capacity, the corresponding mutual information (2) serves as a lower bound to the channel capacity, and is especially meaningful in the analysis of systems where the same codebook must be used under a variety of scattering environments. Expression such as (2) also arise in other scenarios, e.g., in the high SNR analysis of fading channel without receiver side information [2]. Note also that the mutual information (2) is indeed the capacity for a Rayleigh fading channel ($\Lambda = 0$) [3].

[1]This research was conducted while Y. H. Kim was visiting the Swiss Federal Institute of Technology (ETH) Zurich.

Analytic or even numerical calculation of (2) is, however, intractable except for a few special cases. Instead we show that the mutual information (2) is an increasing function of the eigenvalues of Λ, providing a partial solution to an engineering question "Is a larger line-of-sight gain preferable?".

Theorem 1 *Suppose that* $\mathbb{W}$ *and* $\mathbb{W}'$ *are complex non-central Wishart matrices with* $\mathbb{W} \sim \mathcal{W}_n(m, \Lambda)$ *and* $\mathbb{W}' \sim \mathcal{W}_n(m, \Lambda')$ *where* $\Lambda \succeq \Lambda'$ *are nonnegative diagonal matrices with ordered diagonal entries. Then, for all* $\gamma \ge 0$, $\det(\gamma\,\mathsf{I} + \mathbb{W})$ *is stochastically larger than* $\det(\gamma\,\mathsf{I} + \mathbb{W}')$ *and in particular,*

$$\mathsf{E}[\log\det(\gamma\,\mathsf{I} + \mathbb{W})] \ge \mathsf{E}\left[\log\det(\gamma\,\mathsf{I} + \mathbb{W}')\right].$$

The proof is based on the convexity of $\det(\gamma\,\mathsf{I} + \mathsf{X}\mathsf{X}^\dagger)$ in each column of matrix $\mathsf{X} \in \mathcal{C}^{m\times n}$ and the stochastic ordering for multivariate Gaussian distributions over a symmetric convex set [4].

From this theorem it is immediate that, in the presence of receiver-side channel information, the capacity of a Ricean fading channel is no less than that of a Rayleigh fading channel. Combining the Barlett decomposition for Wishart matrices and known results for the expected log determinant of the non-central χ^2 distribution [2], we obtain the following lower bound on the expected log determinant of a Wishart matrix and hence a lower bound on the capacity of a Ricean fading channel with channel state information at the receiver.

Corollary 1 *Let* $\mathbb{W} \sim \mathcal{W}_n(m, \Lambda)$. *Then*

$$g_n(\lambda_1) + \sum_{k=2}^{m}\psi(n-k+1) \le \mathsf{E}[\log\det(\mathbb{W})] \le \sum_{k=1}^{m} g_n(\lambda_k),$$

with equality in the lower bound if, and only if, $\Lambda = \mathrm{diag}(\lambda_1, 0, \ldots, 0)$. *Here*

$$\begin{aligned} g_n(\lambda) &= \log(\lambda) - \mathrm{Ei}(-\lambda) \\ &\quad + \sum_{k=1}^{n-1}\left(\frac{-1}{\lambda}\right)^k\left[e^{-\lambda}(k-1)! - \frac{(n-1)!}{k(n-1-k)!}\right], \end{aligned}$$

ψ *is Euler's digamma function, and* Ei *is the exponential integral function.*

The upper bound follows from Hadamard's inequality.

References

[1] A. T. James, "Distributions of matrix variates and latent roots derived from normal samples," *Ann. Math. Statist.*, vol. 35, no. 2, pp. 475–501, 1964.

[2] A. Lapidoth and S. M. Moser, "Capacity bounds via duality with applications to multi-antenna systems on flat fading channels," submitted to *IEEE Trans. Inform. Theory*, 2002.

[3] İ. E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, 1999.

[4] T. W. Anderson, "The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities," *Proc. Amer. Math. Soc.*, vol. 6, no. 2, pp. 170–176, 1955.

0-7803-7728-1/03/$17.00 ©2003 IEEE.