Approximate Capacity of Index Coding for Some Classes of Graphs

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***Abstract*—For a class of index coding problems with side information graph having the Ramsey number $R(i,j)$ upper bounded by $ci^a j^b$, it is shown that the clique covering scheme approximates the broadcast rate within a multiplicative factor of $O(n^{\frac{a+b}{a+b+1}})$, where n is the number of messages. Based on this result and known bounds on Ramsey numbers, it is demonstrated that the broadcast rate of planar graphs, line graphs, and fuzzy circular interval graphs can be approximated within a factor of $(2n)^{2/3}$.**

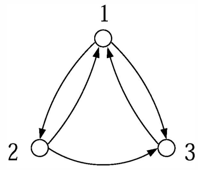

Fig. 1: The graph representation for the index coding problem with $A_1 = \{2,3\}, A_2 = \{1\}$, and $A_3 = \{1,2\}$.

I. Introduction

Index coding is a communication problem in which a server broadcasts n messages $x_1, \ldots, x_n$, $x_j \in \{0,1\}^{t_j}$, to multiple receivers. Suppose that receiver $j \in [n] := \{1,2,\ldots,n\}$ is interested in message x_j and has a subset of other messages $x(A_j) := (x_i, i \in A_j)$, $A_j \subseteq [n] \setminus \{j\}$, as side information. The goal is to find the minimum amount of information that the server needs to broadcast to the receivers such that each receiver can recover its desired message using the broadcast information and its own side information.

Any instance of this problem, referred to collectively as the *index coding problem*, is fully specified by the side information sets $A_1, \ldots, A_n$. Equivalently, it can be specified by a side information graph G with n nodes, in which a directed edge $i \to j$ represents that receiver j has message i as side information, i.e., $i \in A_j$ (see Fig. 1). Thus, we often identify an index coding problem with its side information graph and simply write "index coding problem G."

A (t,r) index code is defined by

- an encoder $\phi : \{0,1\}^{tn} \to \{0,1\}^r$ that maps n-tuple of messages x^n to an r-bit index and
- n decoders $\psi_j : \{0,1\}^r \times \{0,1\}^{t|A_j|} \to \{0,1\}^t$ that maps the received index $\phi(x^n)$ and the side information $x(A_j)$ back to x_j for $j \in [n]$.

Thus, for every $x^n \in \{0,1\}^{tn}$,

$$\psi_j(\phi(x^n), x(A_j)) = x_j, \quad j \in [n].$$

The performance of an index code $\mathcal{C}$ is measured by its rate $\beta(\mathcal{C}) = r/t$. Define the *broadcast rate* of the index coding problem as

$$\beta = \inf_t \inf \beta(\mathcal{C}),$$

where the second infimum is over all (t,r) index codes. Thus, β characterizes the fundamental limit on the rate of index codes such that every message can be recovered exactly. (Sometimes its reciprocal $1/\beta$ is studied as the symmetric capacity.) Despite numerous contributions made over the past two decades, no computable characterization of the broadcast rate of a general index coding problem exists.

For some problems with special structures, however, the broadcast rate has been established (e.g., [1]–[9]). As an example, for the class of (undirected) perfect graphs, the maximum acyclic induced subgraph (MAIS) lower bound is equal to the broadcast rate [1], which is achieved by the clique covering scheme [10] (cf. [9]).

Since characterizing the exact broadcast rate (or capacity) of the index coding problem is hard, both theoretically and computationally, one may attempt to approximate it. For an index coding instance with n messages, it is clear that the broadcast rate β satisfies $1 \le \beta \le n$. Thus uncoded transmission of the messages yields an approximation of the broadcast rate within the trivial multiplicative factor of n. Blasiak, Kleinberg, and Lubetzky [2] proposed an algorithm that approximates the broadcast rate of the general index coding problem within a factor of $O(n \log\log n / \log n)$. Using Ramsey theory, they also showed that for the class of undirected graphs, the clique covering scheme approximates the broadcast rate within a slightly better factor of $O(n/\log n)$. It is unknown, however, how one can approximate the broadcast rate within a factor of $O(n^{1-\epsilon})$, for some $\epsilon > 0$ [2, Sec. VII].

In this paper, we provide a partial answer to the open problem posed by Blasiak, Kleinberg, and Lubetzky. Using the graph theoretic technique developed in [2], we show that if the Ramsey number for a

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class of graphs satisfies $R(i,j) \leq ci^a j^b$ for some constants a, b, and c, then the clique covering scheme approximates the capacity of every n-node graph in that class within a multiplicative factor of $O(n^{\frac{a+b}{a+b+1}})$. The existing literature on Ramsey theory tells us that the classes of planar graphs, line graphs, and fuzzy circular interval graphs satisfy the condition with $a = b = c = 1$, and hence our result implies that for any index coding problem in one of these classes, the clique covering scheme approximates the capacity within a multiplicative factor of $O(n^{2/3})$. Furthermore, we show by the famous four-color theorem [11] that the capacity for the class of planar graphs and complements of planar graphs can be approximated by uncoded transmission and clique covering, respectively, within a factor of four, a significant improvement from the aforementioned approximation based on Ramsey theory. Finally, based on a result on tournaments, we show that uncoded transmission achieves the capacity of the class of *unidirectional* graphs within a factor of $n/\log n$.

The rest of the paper is organized as follows. In Section II, we overview some results from graph theory. In Section III, we present the existing bounds on the capacity of the index coding problem that are used in the paper. In Section IV, we establish a condition under which the capacity of an index coding problem with bidirectional side information graph can be approximated within a multiplicative factor of $O(n^{1-\epsilon})$, for some $\epsilon > 0$. Finally, in Section V, we provide such approximation for the capacity of some classes of directed graphs.

II. Mathematical Preliminaries

Throughout the paper, a graph $G = (V, E)$ (without a qualifier) means a directed, finite, and simple graph, where $V = V(G)$ is the set of vertices and $E = E(G) \subseteq V \times V$ is the set of directed edges. A graph $G = (V, E)$ is said to be *unidirectional* if $(i,j) \in E$ implies $(j,i) \notin E$. Similarly, G is said to be *bidirectional* if $(i,j) \in E$ implies $(j,i) \in E$. Given G, its associated undirected graph $U = U(G)$ is defined by identifying $V(U) = V(G)$ and $E(U) = \{\{i,j\}\colon (i,j) \in E(G) \text{ or } (j,i) \in E(G)\}$. A bidirectional graph G is sometimes identified with its undirected graph. The *complement* $\bar{G}$ of the graph G is defined by $V(\bar{G}) = V(G)$ and $(i,j) \in E(\bar{G})$ if and only (iff) $(i,j) \notin E(G)$. For any $S \subseteq V(G)$, $G|_S$ denotes the subgraph induced by S, i.e., $V(G|_S) = S$ and $E(G|_S) = \{(i,j) \in E\colon i,j \in S\}$.

A. Some Graph Classes

An undirected graph is said to be *planar* if it can be drawn in a plane without graph edges crossing, i.e., edges intersect only at the nodes. Fig. 2 shows an example of a 4-node planar graph and a 5-node graph that is not planar.

The *line* graph of an undirected graph U is obtained by associating a vertex with each edge of the graph U and connecting two vertices with an edge iff the corresponding edges of U have a vertex in common. Fig. 3 shows a graph and its corresponding line graph.

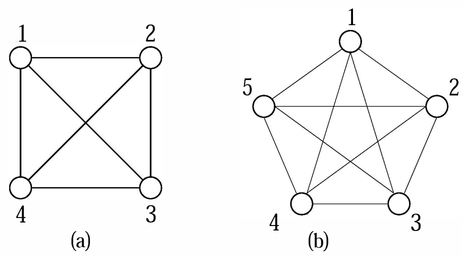


Fig. 2: (a) A 4-node planar graph (edge $\{1,3\}$ can be drawn such that it does not cross $\{2,4\}$). (b) A 5-node non-planar graph.

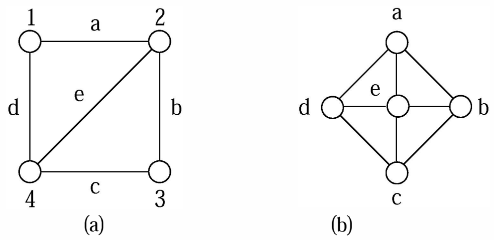


Fig. 3: (a) A 4-node graph with 5 edges. (b) The corresponding 5-node line graph.

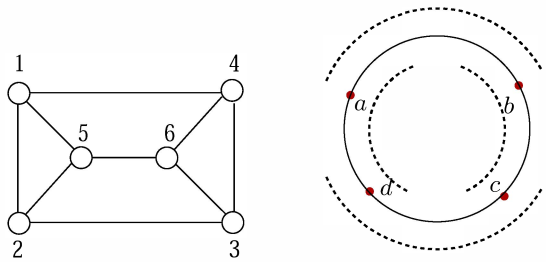
Given a circle C, a closed interval of C is a proper subset of C homeomorphic to the closed unit interval $[0,1]$; in particular, every closed interval of C has two distinct endpoints. An undirected graph $U = (V, E)$ is said to be a *fuzzy circular interval* graph [12] if there exist a set F of closed intervals of a circle C, none including another, such that no point of C is an endpoint of more than one interval in F, and a mapping $\phi : V \to C$ such that if $\{i,j\} \in E$, then $\phi(i)$ and $\phi(j)$ belong to a common interval of F, and if $\{i,j\} \notin E$, then either there is no interval in F that contains both $\phi(i)$ and $\phi(j)$, or there is exactly one interval in F whose endpoints are $\phi(i)$ and $\phi(j)$. As an example, consider the complement of C_6 shown in Fig. 4(a). The corresponding fuzzy circular interval model is shown in Fig. 4(b) where the intervals are shown by dotted arcs and $\phi(1) = \phi(2) = a$, $\phi(3) = \phi(4) = b$, $\phi(5) = d$, and $\phi(6) = c$.

A *tournament* is a unidirectional graph in which every pair of distinct vertices is connected by a single directed edge.

Lemma 1 (Stearns [13], Erdös and Moser [14]). *Every tournament on n vertices contains an acyclic induced subgraph on $1 + \lfloor \log_2 n \rfloor$ vertices.*

B. Graph Coloring

A subset I of the vertices of a graph $G = (V, E)$ is said to be independent if no two vertices of I are adjacent. The maximum size of an independent set of a graph G is referred to as the *independence number* of the graph and is denoted by $\alpha(G)$.



(a) (b)

Fig. 4: (a) The complement of C_6. (b) The fuzzy circular interval model of $\bar{C}_6$ where the intervals are shown by dotted arcs and $\phi(1) = \phi(2) = a$, $\phi(3) = \phi(4) = b$, $\phi(5) = d$, and $\phi(6) = c$.

A (vertex) coloring of an undirected graph U is a mapping that assigns a color to each vertex such that no two adjacent vertices share the same color. The *chromatic number* $\chi(U)$ is the minimum number of colors such that a coloring of the graph exists.

More generally, a b-fold coloring assigns a set of b colors to each vertex such that no two adjacent vertices share the same color. The b-fold chromatic number $\chi^{(b)}(U)$ is the minimum number of colors such that a b-fold coloring exists. The *fractional chromatic number* of the graph is defined as

$$\chi_f(U) = \lim_{b\to\infty} \frac{\chi^{(b)}(U)}{b} = \inf_b \frac{\chi^{(b)}(U)}{b},$$

where the limit exists since $\chi^{(b)}(U)$ is subadditive. Consequently,

$$\chi_f(U) \le \chi(U). \tag{1}$$

Let $\mathcal{I}$ be the collection of all independent sets in U. The chromatic number and the fractional chromatic number are also characterized as the solution to the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \sum_{S\in\mathcal{I}} \rho_S \\ \text{subject to} \quad & \sum_{S\in\mathcal{I}: j\in S} \rho_S \ge 1, \quad j \in V(U). \end{aligned}$$

When the optimization variables ρ_S, $S \in \mathcal{I}$, take integer values $\{0, 1\}$, then the (integral) solution is the chromatic number. If this constraint is relaxed and $\rho_S \in [0, 1]$, then the (rational) solution is the fractional chromatic number [15].

Lemma 2 (Scheinerman and Ullman [15])**.** *For any undirected graph U with n nodes,*

$$\chi_f(U) \ge \frac{n}{\alpha(U)}.$$

The four-color theorem states that the chromatic number of any planar graph is upper bounded by four.

Lemma 3 (Appel, Haken, and Koch [11])**.** *Every planar graph U is four-colorable, i.e., $\chi(U) \le 4$.*

C. Ramsey Numbers

Let $\mathcal{G}$ be a class of undirected graphs, such as perfect graphs and line graphs; see [16] for an overview of common graph classes. Given a class $\mathcal{G}$ of undirected graphs and two positive integers i and j, the *Ramsey number* $R_{\mathcal{G}}(i, j)$ is defined as the smallest positive integer such that every graph in $\mathcal{G}$ with at least $R_{\mathcal{G}}(i, j)$ vertices has a clique of size i or an independent set of size j. If $\mathcal{G}$ is the class of all undirected, finite, and simple graphs, then the Ramsey number is simply denoted by $R(i, j)$. The following lemma presents an upper bound on the Ramsey number.

Lemma 4 (Erdös and Szekeres [17])**.** *For any i, j*

$$R(i, j) \le \binom{i + j - 2}{i - 1}.$$

The following lemma uses Ramsey numbers to indicate a relationship between the independence number of an undirected graph and the chromatic number of its complement.

Lemma 5 (Alon and Kahale [18])**.** *Let $t_k(m) = \max\{j\colon R(k, j) \le m\}$. If $\chi(\bar{U}) \ge n/k + m$, then an independent set of size $t_k(m)$ can be found in U.*

In general, determining Ramsey numbers is very hard and they are known only for very small values of i and j [16]. If either $i \le 2$ or $j \le 2$, then it is straight forward to calculate the Ramsey number $R(i, j)$.

Lemma 6 (Belmonte et al. [16])**.** *For any nonempty graph class $\mathcal{G}$ of undirected graphs,*

$$R_{\mathcal{G}}(1, j) = R_{\mathcal{G}}(i, 1) = 1, \ \ i, j \ge 1.$$

If $\mathcal{G}$ contains all edgeless graphs, then

$$R_{\mathcal{G}}(2, j) = j, \ \ j \ge 1.$$

Similarly, if $\mathcal{G}$ contains all complete graphs, then

$$R_{\mathcal{G}}(i, 2) = i, \ \ i \ge 1.$$

Let $\mathcal{P}$ be the class of planar graphs. As $\mathcal{P}$ contains all edgeless graphs, the Ramsey number for this class is determined by Lemma 6 and the following.

Lemma 7 (Steinberg and Tovey [19])**.**

- $R_{\mathcal{P}}(i, 2) = i, \quad i \le 4, j \ge 1,$
- $R_{\mathcal{P}}(3, j) = 3j - 3, \quad j \ge 1,$
- $R_{\mathcal{P}}(i, j) = 4j - 3, \quad i \ge 4, j \ge 1, (i, j) \ne (4, 2).$

Let $\mathcal{L}$ be the class of line graphs. Since this class contains all edgeless graphs and all complete graphs, Lemma 6 together with the following lemmas determine Ramsey numbers for this class for all pairs (i, j).

Lemma 8 (Matthews and Sumner [20])**.** *For every integer $j \ge 1$, $R_{\mathcal{L}}(3, j) = \lfloor (5j - 3)/2 \rfloor$.*

Lemma 9 (Belmonte, Heggernes, Hof, Rafiey, and Saei [16])**.** *For every pair of integers $i \ge 4$ and $j \ge 1$,*

$$R_{\mathcal{L}}(i, j) = \begin{cases} i(j-1) - (t + r) + 2 & \text{if } i = 2k, \\ i(j-1) - r + 2 & \text{if } i = 2k + 1, \end{cases}$$

where $j = tk + r$, $t \geq 0$ *and* $1 \leq r \leq k$.

Let $\mathcal{F}$ be the class of fuzzy circular interval graphs. This class contains all edgeless graphs and all complete graphs. Hence, Lemma 6 and the following lemma determine Ramsey numbers for the graphs in this class.

Lemma 10 (Belmonte, Heggernes, Hof, Rafiey, and Saei [16]). *For every pair of integers* $i, j \geq 3$,

$$R_{\mathcal{F}}(i,j) = (i-1)j.$$

Therefore, for every pair of integers (i,j), $R_{\mathcal{P}}(i,j)$, $R_{\mathcal{L}}(i,j)$ and $R_{\mathcal{F}}(i,j)$ are determined by Lemmas 6 through 10. Hence, we have a simple bilinear upper bound on the Ramsey numbers of any member of these classes.

Lemma 11. *For every pair of integers* (i,j), $i, j \geq 1$, *and for* $\mathcal{G} = \mathcal{P}, \mathcal{L}$, *or* $\mathcal{F}$, $R_{\mathcal{G}}(i,j) \leq ij$.

III. Existing Bounds on the Broadcast Rate

The simplest approach to index coding is a coding scheme by Birk and Kol [10] that partitions the side information graph G by cliques and transmits the binary sums (parities) of all the messages in each clique.

Proposition 1 (Clique covering bound). *Let* $b_{\mathrm{CC}}(G)$ *be the minimum number of cliques that partition* G, *or equivalently, the chromatic number of* $U(\bar{G})$, *which is the solution to the integer program*

$$\begin{aligned} \textit{minimize} \quad & \sum_{S \in \mathcal{K}} \rho_S \\ \textit{subject to} \quad & \sum_{S \in \mathcal{K}: j \in S} \rho_S \geq 1, \quad j \in V(G), \\ & \rho_S \in \{0,1\}, \quad S \in \mathcal{K}, \end{aligned} \tag{2}$$

where $\mathcal{K}$ *is the collection of all cliques in* G. *Then for any index coding problem* G, $\beta(G) \leq b_{\mathrm{CC}}(G)$.

Blasiak, Kleinberg, and Lubetzky [2] extended this bound to the *fractional clique covering bound*, which is equal to the fractional chromatic number of $U(\bar{G})$, by relaxing the integer constraint $\rho_S \in \{0,1\}$ in (2) to $\rho_S \in [0,1]$. Tighter bounds can be found in [3], [21], [22]. We only need the simpler integral and fractional clique covering bounds for the purpose of this paper.

In [1], Bar-Yossef, Birk, Jayram, and Kol proposed the following lower bound on the broadcast rate of the index coding.

Proposition 2 (Maximum acyclic induced subgraph (MAIS) bound). *For any index coding problem* G

$$\max_{S \subseteq V(G): G|_S \text{ is acyclic}} |S| \leq \beta(G).$$

Remark 1. Since every independent set is acyclic, Proposition 2 implies that for any G, $\alpha(G) \leq \beta(G)$.

IV. Bidirectional Graphs

We begin with the statement of an approximation result for the class of undirected (bidirectional) graphs and its proof, as a similar technique is used to establish the main result of this paper.

Proposition 3 (Blasiak, Kleinberg, and Lubetzky [2]). *For any undirected graph* U *with* n *nodes, the clique covering scheme approximates the broadcast rate of the index coding problem* U *within a factor of* $O(n/\log n)$.

Proof: Let $k = \frac{1}{2}\log n$ and $m = n/k$. Then either $\chi(\bar{U}) < \frac{4n}{\log n}$, or by Lemmas 4 and 5, $\alpha(U) \geq \max\left\{j : R(\frac{1}{2}\log n, j) \leq n/k\right\} \geq \max\left\{j \colon \binom{0.5\log n + j - 2}{0.5\log n - 1} \leq \frac{2n}{\log n}\right\} \geq \frac{1}{2}\log n$, for sufficiently large n. In both cases, $1 \leq \frac{\chi(\bar{U})}{\beta(U)} \leq O(\frac{n}{\log n})$, which completes the proof of the proposition. ■

Next, we present a condition under which there exists an approximation of the broadcast rate within a factor of $O(n^{1-\epsilon})$ for some $\epsilon > 0$.

Theorem 1. *Let* $\mathcal{G}$ *be a class of undirected graphs for which* $R_{\mathcal{G}}(i,j) \leq c i^a j^b$ *holds for some constants* a, b, *and* c. *Then the clique covering scheme approximates the broadcast rate of every* n*-node problem in* $\mathcal{G}$ *within a multiplicative factor of* $2^{\frac{a+1}{a+b+1}} c^{\frac{1}{a+b+1}} n^{\frac{a+b}{a+b+1}}$.

Proof: Let $U \in \mathcal{G}$. Due to Proposition 1, $1 \leq \frac{\chi(\bar{U})}{\beta(U)}$. Let k be a positive real number and consider two cases.
Case 1: If $\chi(\bar{U}) < 2n/k$, then $\frac{\chi(\bar{U})}{\beta(U)} < 2n/k$.
Case 2: If $\chi(\bar{U}) \geq 2n/k$, then

$$\begin{aligned} \left(\frac{n}{ck^{a+1}}\right)^{\frac{1}{b}} &= \max\left\{j \colon ck^a j^b \leq n/k\right\} \\ &\leq \max\left\{j \colon R_{\mathcal{G}}(k,j) \leq n/k\right\} && (3) \\ &= t_k(n/k) && (4) \\ &\leq \alpha(U) && (5) \\ &\leq \beta(U) \leq \chi(\bar{U}) \leq n, \end{aligned}$$

where (3) follows by the assumption of the theorem, and (4) and (5) by letting $m = n/k$ in Lemma 5. Thus,

$$\frac{\chi(\bar{U})}{\beta(U)} \leq \frac{n}{\beta(U)} \leq n^{1-\frac{1}{b}}(ck^{a+1})^{\frac{1}{b}}.$$

As k increases, the upper bound on $\frac{\chi(\bar{U})}{\beta(U)}$ decreases in the first case, and increases in the second case. Hence, to minimize the upper bound on the multiplicative gap between $\chi(\bar{U})$ and $\beta(U)$, we choose $k = 2^{\frac{b}{a+b+1}}(n/c)^{\frac{1}{a+b+1}}$, which makes the upper bounds in both cases to be equal to the desired multiplicative gap. ■

As stated in Lemma 11, planar graphs, line graphs and fuzzy circular interval graphs satisfy the condition of Theorem 1 with $a = b = c = 1$.

Corollary 1. If G is a planar graph or a line graph or a fuzzy circular interval graph with n nodes, the clique covering scheme approximates the broadcast rate within a multiplicative factor of $(2n)^{2/3}$.

V. General Graphs

In [2], Blasiak, Kleinberg, and Lubetzky established a similar result to Proposition 3 for directed graphs.

Proposition 4 (Blasiak, Kleinberg, and Lubetzky [2])**.** *For any index coding problem with n messages, the fractional clique covering scheme approximates the broadcast rate within a multiplicative factor of $O(n \log\log n/\log n)$.*

To the best of our knowledge, the above approximation is the only algorithm to approximate the broadcast rate of a general (directed) index coding problem. In particular, no $O(n^{1-\epsilon})$ approximation exists for any $\epsilon > 0$. In the following, we present such approximation for some classes of directed graphs.

The four-color theorem for planar graphs (Lemma 3) makes it possible to approximate the broadcast rate using simple lower and upper bounds.

Theorem 2. *If either $U(G)$ or $U(\bar{G})$ is planar, the broadcast rate can be approximated within a multiplicative factor of four.*

Proof: If $U(G)$ is planar,

$$\begin{aligned}
\frac{n}{4} &\le \frac{n}{\chi(U(G))} && (6)\\
&\le \frac{n}{\chi_f(U(G))} && (7)\\
&\le \alpha(U(G)) && (8)\\
&\le \beta(U(G)) && (9)\\
&\le \beta(G) \le n, && (10)
\end{aligned}$$

where (6) follows by Lemma 3, (7) by (1), (8) by Lemma 2, (9) by Remark 1, and (10) holds since adding side information decreases the broadcast rate. If $U(\bar{G})$ is planar,

$$1 \le \beta(G) \le \chi(U(\bar{G})) \le 4.$$

■

Due to Theorem 2, if $U(G)$ ($U(\bar{G})$) is planar then uncoded transmission (clique covering) is within a multiplicative factor of four from optimal. Note that Berliner and Langberg [23] showed that for index coding problems with outerplanar side information graph (which is a special case of planar graphs), the best performance over all scalar linear codes is achieved by the clique covering scheme.

Next, consider the class of unidirectional graphs. By Lemma 1, for any unidirectional graph G on n nodes

$$\log n \le \beta(G') \le \beta(G),$$

where G' is a tournament resulting from adding edges with arbitrary direction between the nodes that are not connected in G. This implies that for unidirectional graphs, uncoded transmission is approximately optimal within a factor of $n/\log n$.

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