Problems from the book

Problem 1.1

The number of possible hosts is the number of possible IP addresses, that is, $2^{32}$.

Problem 1.2

If IP addresses are assigned randomly, then a routing table should have an entry for each IP address, that is, $2^{32}$ entries.

Problem 1.4

The transmission of a packet of 1kByte over the link is

$$t_1 = \frac{1\text{kByte}}{100\text{Mbps}} = \frac{8 \times 10^3\text{bits}}{10^8\text{bps}} = 8 \times 10^{-5}\text{s}.$$ 

Let $t_2 = 0.13s$ be the time from the complete transmission of a packet and the complete reception of the ACK. If we wait an ACK after each transmitted packet, then the throughput is

$$R_1 = \frac{1\text{kByte}}{t_1 + t_2} = \frac{8 \times 10^3\text{bits}}{0.13008s} = 61.5\text{kbps}.$$ 

If we wait an ACK after the transmission of every $N$ packets, the throughput becomes

$$R_2 = \frac{N \cdot 8 \times 10^3\text{bits}}{Nt_1 + t_2},$$

that is, $R_2$ tends to 100Mbps (the available rate) as $N \to \infty$.

Problem 2.1

At each time there are 40 active user on average (and they have to share the available 100Mbps). Thus, each user has an average throughput of 2.5Mbps.
**Problem 2.3**

By Little’s result, as $T$ is the average time a packet spends in the system, the average number of packet is

$$L = \lambda T = 1000.$$ 

**Problem 2.4**

The service time is

$$\frac{1}{\mu} = \frac{1kByte}{10Mbps} = \frac{8 \times 10^3 bits}{10^7 bps} = 8 \times 10^{-4}s,$$

and so $\mu = 1250$ packets per second. The arrival rate, in number of packets per second is

$$\lambda_p = \frac{8Mbps}{1kByte} = 10^3.$$

The expected time in the system is

$$T = \frac{1}{\mu - \lambda_p} = \frac{1}{1250 - 1000} = 4ms.$$

The expected time in the queue is

$$T - \frac{1}{\mu} = 3.2ms.$$

**Problem 2.7**

The time a single packet need to be transmitted on a link is

$$t_1 = \frac{1kByte}{10Mbps} = 0.8ms.$$

Assuming that the propagation speed on the fiber is of $5\mu s$ per km (according to the book), the propagation time on each link is $t_2 = 50\mu s$. Let $t_3 = 50\mu s$ be the processing time at each node. The bottleneck is constituted by the transmission rate of a link, so the total time to transmit all $N = 20$ packets is

$$t_{tot} = N \cdot t_1 + 5t_2 + 4t_3 = 20 \cdot 0.8ms + 5 \cdot 0.05ms + 4 \cdot 0.05ms = 16.45ms.$$

**Problem 1**

**M/M/1 Queue**

1) Use the fact above to express $\pi_k$, $k > 0$, as a function of $\pi_0$.

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^k \pi_0$$
2) Using \( \lambda < \mu \) and the fact that all \( \pi_k \)'s sum to 1, compute \( \pi_0 \) (as a function of \( \lambda \) and \( \mu \)).

\[
1 = \sum_{k=0}^{\infty} \pi_k = \pi_0 \sum_{k=0}^{\infty} \left( \frac{\lambda}{\mu} \right)^k = \pi_0 \frac{1}{1 - \lambda/\mu},
\]

because the sum converges as \( \lambda/\mu < 1 \). Hence, we have

\[
\pi_0 = 1 - \lambda/\mu.
\]

3) Using the results above, compute the expected number of packets in the system at any given time. As you learnt in class, you should get \( \lambda/\mu - \lambda \). You may find it useful that \( \rho = \lambda/\mu < 1 \).

Observe that \( \pi_0 = 1 - \rho \). Then, the expected number of packets in the system is

\[
\sum_{k=0}^{\infty} k \pi_k = \pi_0 \sum_{k=0}^{\infty} k \rho^k = \rho \lambda (1 - \rho) \sum_{k=0}^{\infty} k \rho^{k-1}
\]

\[
= \rho (1 - \rho) \sum_{k=0}^{\infty} \frac{\partial}{\partial \rho} \rho^k = \rho (1 - \rho) \frac{\partial}{\partial \rho} \left( \sum_{k=0}^{\infty} \rho^k \right)
\]

\[
= \rho (1 - \rho) \frac{\partial}{\partial \rho} \frac{1}{1 - \rho} = \frac{\rho}{1 - \rho}
\]

\[
= \frac{\lambda}{\mu - \lambda}.
\]

Notice that the sum converges for \( \rho < 1 \), and this allowed to swap the derivative and the sum.

4) What is the expected time \( T_1 \) that a packet spends in the system (queue and service) if the arrival rate is \( \lambda \) and the departure rate is \( 3 \mu \)?

\[
T_1 = \frac{1}{3\mu - \lambda}.
\]

M/M/c Queue

5) What is \( \pi_0 \)?

As before, all the \( \pi \)'s must sum up to 1. Therefore,

\[
1 = \sum_{k=0}^{\infty} \pi_k
\]

\[
= \pi_0 \sum_{k=0}^{c-1} \frac{(cp_c)^k}{k!} + \pi_0 \frac{(cp_c)^c}{c!} \sum_{k=0}^{\infty} \rho^k
\]

\[
= \pi_0 \left( \sum_{k=0}^{c-1} \frac{(cp_c)^k}{k!} + \frac{(cp_c)^c}{c!} \frac{1}{1 - \rho_c} \right)
\]
It follows that
\[ \pi_0 = \left( \sum_{k=0}^{c-1} \frac{(cp_c)^k}{k!} + \frac{(cp_c)^c}{c!} \frac{1}{1 - \rho_c} \right)^{-1} \]

The probability that all servers are occupied (and thus a new arrived packet has to wait in the queue) is
\[ \pi_{c+} = \sum_{k=c}^{\infty} \pi_k. \]

This means that, with probability \( \pi_{c+} \) the new packet has to wait in line, while with probability \( 1 - \pi_{c+} \) it is served right away.

6) Compute \( \pi_{c+} \) as a function of \( c, \rho_c, \pi_0 \).
\[ \pi_{c+} = \sum_{k=c}^{\infty} \pi_k = \frac{(cp_c)^c}{c!} \sum_{k=0}^{\infty} \rho^k = \pi_0 \frac{(cp_c)^c}{c!} \frac{1}{1 - \rho_c}. \]

You could also (but, again, you don’t have to) show that the expected time \( T_c \) that a packet spends in this system (queue and service) is
\[ T_c = \frac{\rho_c}{\lambda(1 - \rho_c)} \pi_{c+}. \]

7) What is the expected time \( T_3 \) that a packet spends in a M/M/3 queue with arrival rate \( \lambda \) and three servers each with service rate \( \mu \)? (Do not write the complete expression for \( \pi_{3+} \))

First compute
\[ \rho_3 = \lambda \frac{3}{3\mu} \]

Then,
\[ T_3 = \frac{\rho_3}{\lambda(1 - \rho_3)} \pi_{3+} = \frac{1}{3\mu - \lambda} \pi_{3+} < \frac{1}{3\mu - \lambda}, \]
as \( \pi_{3+} < 1 \).

8) Compare \( T_3 \) with \( T_1 \) (computed above). For the same price, would you rather buy one outgoing link with rate 30Mbps or three outgoing links each with rate 10Mbps (assuming that you have a box that distributes your outgoing flow between them)?

You would prefer the second choice as the expected delay is smaller.

Problem 2

1) Taking the derivative of \( F_p(x) \), \( p_p(x) = \alpha(x_m)^{\alpha}x^{-\alpha-1} \) for \( x \geq x_m \), and \( p_p(x) = 0 \) for \( x < x_m \).
2) Computing the integral,
\[ \int_{-\infty}^{\infty} p_p(x)dx = \int_{x_m}^{\infty} \alpha(x_m)^{\alpha}x^{-\alpha-1}dx = \alpha(x_m)^{\alpha}(-1/\alpha)((-x_m)^{-\alpha}) = 1. \]
3) \( \bar{F}(x) = \left( \frac{x_m}{x} \right)^\alpha \) for \( x \geq x_m \), \( \bar{F}(x) = 1 \) for \( x < x_m \).

4) 

\[
E[X] = \int_{-\infty}^{\infty} xp_p(x)dx = \alpha(x_m)^\alpha \int_{x_m}^{\infty} x^{-\alpha} dx.
\]

If \( \alpha > 1 \) then the integral is finite, and \( E[X] = \frac{\alpha x_m}{\alpha-1} \). If \( \alpha \leq 1 \) then the integral is infinite, and \( E[X] = \infty \).

5) Consider the density function \( p_p(x) \) for \( x \geq x_m \), and write it as \( p_p(x) = \alpha(x_m/x)^\alpha x^{-1} \). Then

\[
\lim_{\alpha \to \infty} \alpha(x_m/x)^\alpha x^{-1} = \infty
\]

if \( x = x_m \), while

\[
\lim_{\alpha \to \infty} \alpha(x_m/x)^\alpha x^{-1} = 0
\]

for \( x > x_m \). For \( x < x_m \), \( \lim_{\alpha \to \infty} p_p(x) = \lim_{\alpha \to \infty} 0 = 0 \).

Therefore, when \( \alpha \to \infty \), \( p_p(x) \) tends to a Dirac’s delta function centered at \( x_m \).

6) \( \Pr(Z < z) = \Pr(\log(X/x_m) < z) = \Pr(X < x_m e^z) \)

7) Continuing from the question above, \( \Pr(X < x_m e^z) = 1 - \left( \frac{x_m}{x_m e^z} \right)^\alpha = 1 - e^{-z^\alpha} \), that is the Cumulative Distribution Function of an Exponential random variable.