Bode Plots:
Recall the transfer function $H(\omega)$ of an LTI system is such that

$$A \cos(\omega_0 t + \theta) \longrightarrow H(\omega) \longrightarrow A|H(\omega_0)| \cos(\omega_0 t + \angle H(\omega_0) + \theta).$$

$H(\omega)$ represents the change in magnitude and the change in phase when the input to the system is sinusoidal with frequency $\omega$.

Bode plots are a way of plotting and approximating the magnitude $|H(\omega)|$ and phase $\angle H(\omega)$ for a very large range of frequencies without losing precision.

We generally plot the magnitude in decibels:

$$20 \log_{10}(|H(\omega)|)$$

and the phase in radians. In each plot the $\omega$-axis is on a logarithmic scale, i.e. the “ticks” are powers of 10.

Bode plots make use of two mathematical approximations that are fairly accurate when considering orders of magnitude changes in $x$ for $x, a > 0$:

$$\log_{10}\left(\sqrt{1 + \left(\frac{x}{a}\right)^2}\right) \approx \begin{cases} \log_{10}(x) & x \geq a \\ 0 & x < a \end{cases}$$

$$\tan^{-1}\left(\frac{x}{a}\right) \approx \begin{cases} 0 & x < \frac{a}{10} \\ \pi/4 \log_{10}\left(\frac{10x}{a}\right) & \frac{a}{10} \leq x < 10a \\ \pi/2 & x \geq 10a \end{cases}$$

Both of these are approximately linear (in certain regions) on a log scale. i.e.:

$$\log_{10}\left(\sqrt{1 + \left(\frac{x}{a}\right)^2}\right) \approx$$

$$\tan^{-1}\left(\frac{x}{a}\right) \approx$$

We can use these approximations to plot the magnitude and phase of a transfer function $H(\omega)$. 

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Suppose we have a transfer function

\[
H(\omega) = \frac{x_0 + x_1(j\omega) + x_2(j\omega)^2 + \cdots + x_m(j\omega)^m}{y_0 + y_1(j\omega) + y_2(j\omega)^2 + \cdots + y_n(j\omega)^n}
\]

where \(x_0, \ldots, x_m\) and \(y_0, \ldots, y_n\) are complex numbers. (Nearly all transfer functions we encounter are of this form). By factoring polynomials, we can write \(H(\omega)\) in **standard form**:

\[
H(\omega) = C \omega^k \left( \frac{1 + \frac{j\omega}{a_1}}{1 + \frac{j\omega}{b_1}} \right) \cdots \left( \frac{1 + \frac{j\omega}{a_r}}{1 + \frac{j\omega}{b_s}} \right)
\]

where \(a_1, \ldots, a_r\) and \(b_1, \ldots, b_s\) are real numbers, \(C\) is some constant complex number, and \(k\) is some positive or negative integer. Having a transfer function in **standard form** allows us to use our earlier approximations, since we can write each linear term as

\[
1 + \frac{j\omega}{a} = \sqrt{1 + \left( \frac{\omega}{a} \right)^2} e^{j\tan^{-1}(\frac{\omega}{a})}
\]

**Approximating the magnitude of a transfer function in standard form:**

The magnitude of a product of complex numbers is the product of the magnitudes (i.e. \(|XY| = |X||Y|\)), and the log of a product is the sum of the logs (i.e. \(\log(AB) = \log(A) + \log(B)\)). Thus

\[
20 \log_{10}(|H(\omega)|) = 20 \left( \log_{10}(|C|) + k \log_{10}(\omega) + \sum_{n=1}^{r} \log_{10} \left( \sqrt{1 + \left( \frac{\omega}{a_n} \right)^2} \right) - \sum_{n=1}^{s} \log_{10} \left( \sqrt{1 + \left( \frac{\omega}{b_n} \right)^2} \right) \right)
\]

\[
20 \log_{10} \left( \sqrt{1 + \left( \frac{\omega}{a_n} \right)^2} \right) \approx \begin{cases} 
0 & \text{when } \omega < a_n \\
\text{adds 20 dB/dec} & \text{when } \omega \geq a_n
\end{cases}
\]

20 \(\log_{10}(|C|)\) is present at all frequencies.

20 \(\log_{10}(\omega)\) is zero when \(\omega = 1\) and adds 20 \(k\) dB/dec.

**Approximating the phase of a transfer function in standard form:**

The phase of a product of complex numbers is the sum of the phases (i.e. \(\angle XY = \angle X + \angle Y\)). Thus

\[
\angle H(\omega) = \angle(C) + \angle(\omega^k) + \sum_{n=1}^{r} \tan^{-1} \left( \frac{\omega}{a_n} \right) - \sum_{n=1}^{s} \tan^{-1} \left( \frac{\omega}{b_n} \right)
\]

\(\angle(C)\) is present at all frequencies, and \(\omega^k\) is a real number, so \(\angle(\omega^k) = 0\).

\[
\tan^{-1} \left( \frac{\omega}{a_n} \right) \approx \begin{cases} 
0 & \text{when } \omega < a_n/10 \\
\pi/4 \text{ rads/dec} & \text{when } a_n/10 \leq \omega < 10a_n \\
\pi/2 & \text{when } \omega \geq 10a_n
\end{cases}
\]
Approximating Values from the Bode Plot:
We can use the fact Bode plots are a logarithmic linear approximation of the magnitude and phase to find the values of points:

\[
\text{Slope } = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{\log(x_2) - \log(x_1)} = \frac{y_2 - y_1}{\log(x_2/x_1)}.
\]

Example 1:
Draw the Bode plot (amplitude and phase) and find all critical points of the transfer function

\[H(\omega) = \frac{j\omega}{1000 + j\omega}\]

\(H(\omega)\) is a simple high pass filter. We first write \(H(\omega)\) in standard form:

\[H(\omega) = \frac{j\omega}{1000(1 + \frac{j\omega}{1000})}\]

Then

\[
20 \log_{10}(|H(\omega)|) = 20 \log_{10}(\omega) - 60 - 20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega}{1000}\right)^2}\right)
\]

\[\angle H(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega}{10^3}\right).
\]
To find the height of point A, we note $20 \log_{10}(1) = 0$, so $-60$ is the only “active” term at $\omega = 1$. To find the height of point B, we use the fact the slope is $20$ dB per decade. So the height of B is

$$-60 + 20(\log_{10}(10^3) - \log_{10}(1)) = 0.$$  

We also have $C = (100, \pi/2)$ and $D = (10^4, 0)$.

**Example 2:**

*Draw the Bode plot (amplitude and phase) and find all critical points of the transfer function*

$$H(\omega) = \frac{500 + 5j\omega}{(1 + j\omega)(10 + \frac{j\omega}{100})}$$

We first write $H(\omega)$ in standard form:

$$H(\omega) = \frac{50(1 + \frac{j\omega}{100})}{(1 + j\omega)(1 + \frac{j\omega}{1000})}$$

Then

$$20 \log_{10}(|H(\omega)|) = 20 + 20 \log_{10}(5) + 20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega}{100}\right)^2}\right) - 20 \log_{10}\left(\sqrt{1 + \omega^2}\right) - 20 \log_{10}\left(\sqrt{1 + \left(\frac{\omega}{1000}\right)^2}\right)$$

$$\angle H(\omega) = \tan^{-1}\left(\frac{\omega}{100}\right) - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{1000}\right).$$

The height at point A is $20(1 + \log_{10}(5)).$

The height at point B is $-20 + 20 \log_{10}(5)$, since the slope is $-20$ dB/decade and the frequency changes by a factor of 100.

$20 \log_{10}(|H(\omega_0)|) = 0$ when

$$0 = 20(1 + \log_{10}(5)) - 20(\log_{10}(\omega_0) - \log_{10}(1))$$

$$\rightarrow 1 + \log_{10}(5) = \log_{10}(\omega_0) \rightarrow \omega_0 = 50$$
Example 3:

Draw the Bode plot (amplitude and phase) and find all critical points of the transfer function

\[ H(\omega) = \frac{3 (400 - \omega^2 + 40j\omega)}{2 (j\omega + 6000)(j\omega \times 10^{-6} + 1)} \]

We first write \( H(\omega) \) in standard form:

\[ H(\omega) = \frac{(\frac{j\omega}{20} + 1)^2}{10 (\frac{j\omega}{6000} + 1) (\frac{j\omega}{10^6} + 1)} \]

Then

\[ 20 \log_{10}(|H(\omega)|) = 40 \log_{10} \left( \sqrt{1 + \left(\frac{\omega}{20}\right)^2} \right) - 20 - 20 \log_{10} \left( \sqrt{1 + \left(\frac{\omega}{6000}\right)^2} \right) - 20 \log_{10} \left( \sqrt{1 + \left(\frac{\omega}{10^6}\right)^2} \right) \]

\[ \angle H(\omega) = 2 \tan^{-1} \left(\frac{\omega}{20}\right) - \tan^{-1} \left(\frac{\omega}{6000}\right) - \tan^{-1} \left(\frac{\omega}{10^6}\right) \].

The change in \( \log(\omega) \) from point A to B is \( \log(6000) - \log(20) \), and the slope (dB per decade) in this region is 40, so the height at B is

\[ -20 + 40(\log(6000) - \log(20)) = 60 + 40 \log(3) \].

The height at C is

\[ (60 + 40 \log(3)) + 20(\log(10^6) - \log(6000)) = 120 + 20 \log(3) - 20 \log(2) \].
\[ 20 \log_{10}(|H(\omega_0)|) = 0 \text{ when} \]
\[ 0 = -20 + 40(\log(\omega_0) - \log(20)) \quad \rightarrow \quad \omega_0 = 20 \sqrt{10} \approx 63.2 \]

We also have

\[ D = (2, 0), \; E = (200, \pi), \; F = (600, \pi), \; G = (60000, \pi/2), \; H = (10^5, \pi/2), \; I = (10^7, 0) \]