ECE 45 Discussion 5 Solutions

Topics

• Fourier Transform

Fourier Transform

Prior to now, our analysis of linear systems (circuits) has been limited to periodic functions. How can we analyze outputs of systems that are not periodic? We use the Fourier transform to represent a function as an integral of sinusoidal components—similar to the Fourier series summation representation of a periodic signal. The Fourier transform is more or less a Fourier series with infinite period (thus \( \omega_0 \rightarrow 0 \)).

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t} d\omega
\]

\[
F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt
\]

The components of the Fourier series, \( F_n \), each represented the contribution from the sinusoid with frequency \( n\omega_0 \). The components of the Fourier transform are represented as a continuous function, rather than discrete points, i.e. rather than only having contributions from sinusoids at frequencies at multiples of the fundamental frequency, non-periodic functions can have contributions from sinusoids at ALL frequencies.

Fourier Transform as an Input to an LTI System:

Since a Fourier transform is an integral (sum with infinitesimally small intervals) of sinusoidal components, we know how to analyze the output of a system with a periodic function as its input. For any input \( x(t) \).

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega \rightarrow H(\omega) \rightarrow y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|X(j\omega)e^{j(\omega t + \angle H(\omega))} d\omega
\]

We can represent \( y(t) \) as a Fourier Transform:

\[
Y(j\omega) = X(j\omega) H(\omega)
\]

The Fourier transform is a one-to-one mapping, so if we know either the time function or the frequency function, we know the other as well. It is often convenient or useful to go back and forth between the time and frequency representations of functions.

Other applications:

Our main use of the FT right now is for linear system analysis, but this extends to many different applications. If we are transmitting or broadcasting data, we may care about the bandwidth that we use. By looking at the Fourier transform of a signal, we can see what range of frequencies contribute to it.

Different materials respond differently to different frequency signals (i.e. water essentially acts as a LPF, blocking high frequencies and allowing low). By knowing what frequencies our signal spans, we can determine if our signal will deteriorate.
Example 1: Determine the Fourier transform, $F(j\omega)$, of the signal $f(t) = e^{-a|t|}$ where $a > 0$

$$F(j\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}$$

Example 2: Determine the time representation of $H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & \text{else} \end{cases}$

Note: $H(j\omega)$ is an ideal low-pass filter

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega = \frac{1}{2\pi} \frac{e^{j\omega_0 t}}{jt} \bigg|_{\omega=-\omega_0}^{\omega=\omega_0} = \frac{1}{\pi t} \left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)$$

$$= \frac{1}{\pi t} \sin(\omega_0 t) = \frac{\omega_0}{\pi} \frac{\sin(\omega_0 t)}{\omega_0 t} = \frac{\omega_0}{\pi} \text{sinc}(\omega_0 t)$$

Example 3: Find the output $y(t)$ when $x(t) = \text{rect}(\frac{t}{2})$ and $H(\omega) = 2e^{-j4\omega}$

Note: $A\text{ rect}(\frac{t-t_0}{W})$ is a rectangular pulse with height $A$, width $W$, and centered at $t_0$.

$$A\text{ rect}(\frac{t-t_0}{W}) = \begin{cases} A & (t_0 - W/2) \leq t \leq (t_0 + W/2) \\ 0 & \text{else} \end{cases}$$

$$x(t) = \begin{cases} 1 & |t| \leq 1 \\ 0 & \text{else} \end{cases}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-1}^{1} e^{-j\omega t} dt = \frac{1}{-j\omega} (e^{-j\omega} - e^{j\omega}) = \frac{2}{\omega} \left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right) = \frac{2}{\omega} \sin(\omega)$$

$$X(j\omega) = 2 \text{sinc}(\omega)$$

$$Y(j\omega) = X(j\omega)H(\omega) = 4e^{-j4\omega} \text{sinc}(\omega)$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 4e^{-j4\omega} \text{sinc}(\omega) e^{j\omega t} d\omega = 2 \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \text{sinc}(\omega) e^{j\omega(t-4)} d\omega\right)$$

let $t' = t - 4$

$$y(t) = 2 \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \text{sinc}(\omega) e^{j\omega t'} d\omega\right) = 2 \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t'} d\omega\right) = 2 x(t')$$

$$= 2 x(t - 4) = 2 \text{ rect}\left(\frac{t - 4}{2}\right)$$
Example 4: Find the output $y(t)$ when $x(t) = 1$ and $H(\omega) = j\omega$

We could approach this problem as we did the last, but instead, let’s use our knowledge of linear systems to save us some work.

We know that $Y(j\omega) = X(j\omega)H(\omega) = X(j\omega)j\omega$. We don’t know what $X(j\omega)$ is off-hand, but we do know that multiplying by $j\omega$ corresponds to taking a time derivative.

$$y(t) = \frac{dx(t)}{dt} = \frac{d}{dt}(1) = 0$$

Example 5: If $x(t) = e^{-2|t|}$ and $y(t) = 0.5 e^{-2|t-1|}$ What is $H(\omega)$?

Notice $y(t) = 0.5 x(t - 1)$

$$y(t) = 0.5 x(t - 1) = 0.5 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-1)} d\omega \right)$$

$$= 0.5 \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} (X(j\omega) e^{-j\omega}) e^{j\omega t} d\omega \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega) e^{j\omega t} d\omega$$

Thus $Y(j\omega) = 0.5 X(j\omega) e^{-j\omega}$

$$H(\omega) = \frac{Y(j\omega)}{X(j\omega)} = 0.5 e^{-j\omega}$$

We didn’t even need to calculate $X(j\omega)$ and $Y(j\omega)$ (although we could have, and we would have gotten the same result as above).