Fourier Series Properties

We can represent a periodic function as a sum of sinusoidal components with different coefficients, i.e. a Fourier Series. However, the calculations of the coefficients is often tedious and time-consuming. We can use the properties of the Fourier series to simplify calculations of similar signals. Proving each of these properties is a good exercise.

Fourier Series as an Input to an LTI System

A Fourier series is a sum of sinusoidal components. We know how to analyze LTI systems for sinusoidal inputs, so by linearity, we can determine the output when we have a periodic input. Let \( x(t) \) be periodic with fundamental frequency \( \omega_0 \)

\[
x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j\omega_0 nt} \rightarrow H(\omega) \rightarrow y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) X_n e^{j\omega_0 nt}
\]

Notice that \( y(t) \) is also a periodic function with fundamental frequency \( \omega_0 \). We can represent using a Fourier series with coefficients:

\[
Y_n = X_n H(n\omega_0)
\]

Linearity and Scaling

If we have two periodic functions \( f(t) \) and \( g(t) \) with fundamental frequency \( \omega_0 \) and coefficients \( F_n \) and \( G_n \) respectively.

Let \( h(t) = a f(t) + b g(t) \)

\( h(t) \) will also be periodic with fundamental frequency \( \omega_0 \) and coefficients:

\[
H_n = a F_n + b G_n
\]

Time Reversal

If we have two periodic functions \( f(t) \) and \( g(t) \) such that \( g(t) = f(-t) \), then \( G_n = F_{-n} \).

Time Shifting

If we have two periodic functions \( f(t) \) and \( g(t) \) such that \( g(t) = f(t - t_0) \), then \( G_n = F_n e^{-j\omega_0 n t_0} \).

Time Scaling

If we have two periodic functions \( f(t) \) and \( g(t) \) such that \( g(t) = f(at) \) and the fundamental frequency of \( f(t) \) is \( \omega_0 \), then \( G_n = F_n \), but the fundamental frequency of \( g(t) \) is \( a\omega_0 \).
**Time Derivative**

If we have two periodic functions $f(t)$ and $g(t)$ such that $g(t) = \frac{df(t)}{dt}$, then $G_n = j\omega_0 n F_n$.

**Time Multiplication**

If we have three periodic functions $f(t)$, $g(t)$, and $h(t)$ such that $h(t) = f(t) g(t)$, the coefficients for $h(t)$ can be determined from the coefficients for $f(t)$ and $g(t)$

$$H_n = \sum_{k=-\infty}^{\infty} F_k G_{n-k}$$

**Parseval’s Theorem**

The *average power in a period* of a periodic function can be calculated two ways:

$$P_{avg} = \frac{1}{T} \int_T |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Sometimes the calculations for one method are much simpler than the other.

**Real Functions**

If $f(t)$ is a real function, then $F_n = F^*_{-n}$. The Fourier series of a real function can be simplified to a sum of sines and cosines.

This follows from the fact that for a real function $f(t) = f^*(t)$.

**Imaginary Functions**

If $f(t)$ is an imaginary function, then $F_n = -F^*_{-n}$

This follows from the fact that for a real function $f(t) = -f^*(t)$.

**Even Functions**

If $f(t)$ is an even function, then $F_n = F_{-n}$. The Fourier series of an even function can be simplified to a sum of cosines.

This follows from the fact that for an even function $f(t) = f(-t)$.

**Odd Functions**

If $f(t)$ is an odd function, then $F_n = -F_{-n}$. The Fourier series of an odd function can be simplified to a sum of sines.

This follows from the fact that for an even function $f(t) = -f(-t)$. 


Examples

1. Write \( f(t) \) as a sum of sines and cosines and find the average power in a period, where

We note that \( f(t) \) has period 2 and in the period \([-1, 1), f(t) = At\). So for all \( n \neq 0 \), we have

\[
F_n = \frac{1}{T} \int_T f(t) e^{-j\omega_0 nt} dt = \frac{A}{2} \int_{-1}^1 t e^{-j\pi nt} dt
\]

\[
= \frac{A}{-j2\pi n} \left( te^{-j\pi nt}|_{-1}^1 - \int_{-1}^1 e^{-j\pi nt} dt \right)
\]

\[
= \frac{-A}{j2\pi n} \left( e^{-j\pi n} + e^{j\pi n} + \frac{1}{j\pi n} (e^{-j\pi n} - e^{j\pi n}) \right) = \frac{-A}{j\pi n} (-1)^n.
\]

In the second line, we divide by 0 when \( n = 0 \), so

\[
F_0 = \frac{A}{2} \int_{-1}^1 t e^0 dt = 0.
\]

Thus we have

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_0 nt} = \sum_{n=-\infty}^{\infty} \frac{-A}{j\pi n} (-1)^n e^{j\pi n} - \frac{-A}{j\pi n} (-1)^{-n} e^{-j\pi nt}
\]

\[
= -A \sum_{n=1}^{\infty} (-1)^n \left( \frac{e^{j\pi nt} - e^{-j\pi nt}}{j\pi n} \right) = -2A \sum_{n=1}^{\infty} (-1)^n \sin(\pi nt)
\]

The average power in a period is a straight-forward calculation in the time domain:

\[
\frac{1}{2} \int_{-1}^1 f(t)^2 dt = \frac{A^2}{2} \int_{-1}^1 t^2 dt = \frac{A^2}{3}.
\]

By Parseval’s Theorem:

\[
\frac{A^2}{3} = \sum_{n=-\infty}^{\infty} |F_n|^2 = \sum_{n=-\infty}^{\infty} \left| \frac{-A}{j\pi n} (-1)^n \right|^2 = \sum_{n=-\infty}^{\infty} \frac{A^2}{\pi^2 n^2} = \sum_{n=1}^{\infty} \frac{2A^2}{\pi^2 n^2}
\]

Thus we have

\[
\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

which is a beautiful bit of mathematics :)
2. Write \( g(t) \) as a sum of sines and cosines and find the Fourier series components \( G_n \), where

\[
g(t) = 1 + 2A \sum_{n=1}^{\infty} (-1)^n \sin(\pi n(2t - 2)) = 1 + A \sum_{n=1}^{\infty} (-1)^n \sin(2\pi nt).
\]

Note that we have \( g(t) = -f(2t - 2) + 1 \), so

In order to find \( G_n \) in terms of \( F_n \), let’s use some intermediate steps:

Let \( x(t) = -f(t - 1) \), then \( x(t) \) is periodic with period 2 and \( X_n = -F_n e^{-j\pi n} \).

Let \( y(t) = x(2t) \), then \( y(t) \) is periodic with period 1 and \( Y_n = X_n \).

Then \( g(t) = y(t) + 1 \), so for \( n \neq 0 \),

\[
G_n = -F_n(-1)^n = \frac{A}{j\pi n}
\]

and \( G_0 = F_0 + 1 = 1 \).

3. Suppose \( g(t) \) is the input to an LTI system with frequency response \( H(\omega) = j\omega \). Find the output \( z(t) \). Approximately what type of waveform is \( z(t) \)?

Since \( g(t) \) is periodic with period 1, \( z(t) \) is also periodic with period 1 and the Fourier series coefficients of \( z(t) \) are given by

\[
Z_n = H(\omega_0 n)G_n = j\omega_0 n G_n
\]

which implies

\[
\frac{d}{dt}g(t) = A.
\]

4. Suppose \( g(t) \) is the input to an LTI system with frequency response \( H(\omega) = 2 \), when \( |\omega| > \pi \) and is 0 otherwise. Plot the output \( w(t) \).

Since \( g(t) \) is periodic with period 1, \( w(t) \) is also periodic with period 1 and the Fourier series coefficients of \( w(t) \) are given by

\[
W_n = H(\omega_0 n)G_n = H(2\pi n)G_n = \left\{
\begin{array}{ll}
G_n & \text{if } |2\pi n| > \pi \\
0 & \text{otherwise}
\end{array}
\right.
\]

So we have \( W_n = G_n \) for all \( n \neq 0 \) and \( W_0 = 0 \), which implies \( w(t) = g(t) - G_0 = g(t) - 1 \).