Solutions to Final Examination

1. Noisy forecasters (60 points).
Suppose that tomorrow’s weather is modeled by \( X \sim \text{Bern}(1/2) \), namely, it is rainy (i.e., \( X = 1 \)) with probability 1/2 and sunny (i.e., \( X = 0 \)) with probability 1/2. There are three weather forecasters—Alice, Bob, and Charlie—who predict the weather as \( A, B, \) and \( C \), respectively, each with 1/4 probability of error. In other words, their predictions are the outputs of the binary symmetric channel (BSC) with input \( X \) and crossover probability \( p = 1/4 \). We assume that their prediction errors are independent; thus

\[
p_{A,B,C|X}(a, b, c|x) = p_{A|X}(a|x)p_{B|X}(b|x)p_{C|X}(c|x)
\]

for \( x, a, b, c \in \{0, 1\} \). Let

\[
Y = A + B + C
\]
denote the number of forecasters who predict that it will be rainy tomorrow.

(a) Find the conditional pmf of \( Y \) given \( X \).
(b) Find the pmf of \( Y \).
(c) Find the conditional pmf of \( X \) given \( Y \).
(d) Find the conditional pmf of \( X \) given \( \{Y \leq 1\} \).
(e) Find the decision rule \( d(y) \) that minimizes the probability of error \( P\{X \neq d(Y)\} \).
(f) Find the associated probability of error.

Solution:

(a) We have

\[
p_{Y|X}(0|0) = P\{Y = 0|X = 0\}
= P\{A = 0, B = 0, C = 0|X = 0\}
= p_{A|X}(0|0)p_{B|X}(0|0)p_{C|X}(0|0)
= (p_{A|X}(0|0))^3
= \left(\frac{3}{4}\right)^3 = \frac{27}{64}.
\]

Similarly, we have

\[
p_{Y|X}(0|0) = p_{Y|X}(3|1) = 27/64,
p_{Y|X}(1|0) = p_{Y|X}(2|1) = 27/64,
p_{Y|X}(2|0) = p_{Y|X}(1|1) = 9/64,
p_{Y|X}(3|0) = p_{Y|X}(0|1) = 1/64.
\]
(b) From part (a), we have
\[ p_Y(0) = p_{Y|X}(0|0)p_X(0) + p_{Y|X}(0|1)p_X(1) = \frac{7}{32}. \]
Similarly, we have \( p_Y(1) = p_Y(2) = \frac{9}{32} \) and \( p_Y(3) = \frac{7}{32}. \)

(c) By the Bayes’ rule, we have
\[ p_{X|Y}(0|0) = \frac{p_{Y|X}(0|0)p_X(0)}{p_Y(0)} = \frac{27}{28}. \]
Similarly, we have
\[ p_{X|Y}(0|0) = \frac{27}{28} \quad \text{and} \quad p_{X|Y}(1|0) = \frac{1}{28}, \]
\[ p_{X|Y}(0|1) = \frac{3}{4} \quad \text{and} \quad p_{X|Y}(1|1) = \frac{1}{4}, \]
\[ p_{X|Y}(0|2) = \frac{1}{4} \quad \text{and} \quad p_{X|Y}(1|2) = \frac{3}{4}, \]
\[ p_{X|Y}(0|3) = \frac{1}{28} \quad \text{and} \quad p_{X|Y}(1|3) = \frac{27}{28}. \]

(d) Again, by the Bayes’ rule, we have
\[
P\{X = 0|Y \leq 1\} = \frac{P\{Y \leq 1|X = 0\}P\{X = 0\}}{P\{Y \leq 1\}}
= \frac{P\{Y = 0|X = 0\}P\{X = 0\} + P\{Y = 1|X = 0\}P\{X = 0\}}{P\{Y = 0\} + P\{Y = 1\}}
= \frac{27}{32}
\]
and
\[
P\{X = 1|Y \leq 1\} = \frac{5}{32}.
\]

(e) Since \( P\{X = 0\} = P\{X = 1\} = \frac{1}{2} \), the MAP rule becomes the ML rule. Therefore,
\[
d(y) = \begin{cases} 
0 & \text{for } y = 0 \text{ or } 1, \\
1 & \text{for } y = 2 \text{ or } 3.
\end{cases}
\]

(f) The associated probability of error is
\[
P_e = P\{X \neq d(Y)\}
= P\{X \neq d(Y)|X = 0\}P\{X = 0\} + P\{X \neq d(Y)|X = 1\}P\{X = 1\}
= P\{Y = 2 \text{ or } 3|X = 0\}P\{X = 0\} + P\{Y = 0 \text{ or } 1|X = 1\}P\{X = 1\}
= \frac{5}{32}.
\]
2. Estimation (20 points).
Let $X_1, X_2, X_3$ be i.i.d. random variables with finite mean and variance. Let $Y = X_1 + X_2 + X_3$.

(a) Find the MMSE estimate of $X_1$ given $Y$.
(b) Find the MMSE estimate of $X_1 + 2X_2$ given $Y$.

Solution:
(a) This problem is very similar to Problem 3 in the midterm. First note that by symmetry
$$E(X_1|Y) = E(X_2|Y) = E(X_3|Y).$$
Furthermore,
$$Y = E(X_1 + X_2 + X_3|Y) = E(X_1|Y) + E(X_2|Y) + E(X_3|Y) = 3E(X_1|Y).$$
Hence, $E(X_1|Y) = \frac{1}{3}Y$. The MMSE estimate of $X_1$ given $Y$ is
$$E(X_1|Y) = \frac{1}{3}Y.$$
(b) Similarly, the MMSE estimate of $X_1 + 2X_2$ given $Y$ is
$$E(X_1 + 2X_2|Y) = E(X_1|Y) + 2E(X_2|Y) = 3E(X_1|Y) = Y.$$

3. Moving average process (80 points)
Let
$$X_n = Z_{n-1} + Z_n, \quad n = 1, 2, \ldots,$$
where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0, 1)$. Let
$$Y_n = X_{n-1} - X_n, \quad n = 2, 3, \ldots.$$

(a) Find the mean and autocorrelation functions of $\{Y_n\}$.
(b) Is $\{Y_n\}$ wide-sense stationary? Justify your answer.
(c) Is $\{Y_n\}$ strict-sense stationary? Justify your answer.
(d) Is $\{Y_n\}$ Markov?
(e) Is $\{Y_n\}$ independent-increment?
(f) Find $E(Y_2|Y_3)$.
(g) Find $E(Y_2|Y_4)$.
(h) Find $E(Y_2|Y_3, Y_4)$.

Solution: See the last page.

4. Random-delay mixture (40 points).
Let $\{X(t)\}, -\infty < t < \infty$, be a zero-mean wide-sense stationary process with autocorrelation function $R_X(\tau) = e^{-|\tau|}$. Let
$$Y(t) = X(t - U),$$
where $U$ is a random delay, independent of $\{X(t)\}$. Suppose that $U \sim Bern(1/2)$, that is,
$$Y(t) = \begin{cases} X(t) & \text{with probability } 1/2, \\ X(t-1) & \text{with probability } 1/2. \end{cases}$$
(a) Find the mean and autocorrelation functions of \( \{Y(t)\} \).

(b) Is \( \{Y(t)\} \) wide-sense stationary? Justify your answer.

(c) Find the average power \( \mathbb{E}(Y(t)^2) \) of \( \{Y(t)\} \).

(d) Now suppose that \( U \sim \text{Exp}(1) \), i.e., \( f_U(u) = e^{-u}, \; u \geq 0 \). Find the autocorrelation function of \( \{Y(t)\} \).

Solution:

(a) By the iterated expectation and the independence of \( U \) and \( \{X(t)\} \), we have

\[
\mathbb{E}(Y(t)) = \mathbb{E}(X(t - U)) \\
= \mathbb{E}[\mathbb{E}(X(t - U)|U)] \\
= \frac{1}{2} \mathbb{E}(X(t)|U = 0) + \frac{1}{2} \mathbb{E}(X(t - 1)|U = 1) \\
= \frac{1}{2} \mathbb{E}(X(t)) + \frac{1}{2} \mathbb{E}(X(t - 1)) \\
= 0
\]

and

\[
R_Y(t_1, t_2) = \mathbb{E}(Y(t_1)Y(t_2)) \\
= \mathbb{E}(X(t_1 - U)X(t_2 - U)) \\
= \mathbb{E}[\mathbb{E}(X(t_1 - U)X(t_2 - U)|U)] \\
= \frac{1}{2} \mathbb{E}(X(t_1)X(t_2)|U = 0) + \frac{1}{2} \mathbb{E}(X(t_1 - 1)X(t_2 - 1)|U = 1) \\
\overset{(a)}{=} \mathbb{E}(X(t_1)X(t_2)) \\
= e^{-|t_1 - t_2|}
\]

where (a) follows by the stationarity of \( \{X(t)\} \).

(b) Since \( \mathbb{E}(Y(t)) \) and \( R_Y(t_1, t_2) \) are time invariant, it is WSS.

(c) From part (a), we have

\[
\mathbb{E}(Y(t)^2) = R_X(0) = 1.
\]

(d) For \( U \sim \text{Exp}(1) \), we have

\[
\mathbb{E}(Y(t)) = \mathbb{E}(X(t - U)) \\
= \mathbb{E}[\mathbb{E}(X(t - U)|U)] \\
= \int_0^\infty \mathbb{E}(X(t - u))e^{-u}du \\
= 0
\]
and

\[ R_Y(t_1, t_2) = \mathbb{E}(Y(t_1)Y(t_2)) \]
\[ = \mathbb{E}(X(t_1 - U)X(t_2 - U)) \]
\[ = \mathbb{E}[\mathbb{E}(X(t_1 - U)X(t_2 - U)|U)] \]
\[ = \int_0^\infty \mathbb{E}(X(t_1 - u)X(t_2 - u)e^{-u} du \]
\[ = \int_0^\infty R_X(t_1 - t_2)e^{-u} du \]
\[ = R_X(t_1 - t_2) \]
\[ = e^{-|t_1 - t_2|}. \]

Note that \( \{Y(t)\} \) is WSS with \( R_Y(t_1, t_2) = R_X(t_1, t_2) \) for any random delay \( U \). In fact, \( \{Y(t)\} \) has the same distribution as \( \{X(t)\} \), not only the first and second moments.
Correction of Problem 3

a. \[ R_X(n) = \begin{cases} \frac{2}{3} & , n = 0 \\ \frac{1}{3} & , n = \pm 1 \\ 0 & \text{otherwise} \end{cases} \]

d. No,
\[ Y_n = Z_{n-2} - Z_n, \quad Y_{n-1} = Z_{n-3} - Z_{n-1}, \quad Y_{n-2} = Z_{n-4} - Z_{n-2} \]
The probability distribution of \( Y_n | Y_{n-1}, Y_{n-2} \) is equal to that of \( Y_n | Y_{n-1} \), since \( Y_{n-1} \) is independent of \( Y_n \) and \( Y_{n-2} \).

e. \( Y \) is the output of \( X \) through the LTI system \( h(i) = \begin{cases} 1, & i = 1 \\ -1, & i = 0 \\ 0, & \text{o.w.} \end{cases} \)

As a result, \( Y \) is also WSS and then apply the argument of the solution.

f. Since \((Z_0, Z_2)\) and \((Z_1, Z_3)\) are independent, \( Y_2 = Z_0 - Z_2 \) and \( Y_3 = Z_1 - Z_3 \) are independent. \[ E(Y_2 | Y_3) = E(Y_2) = E(Z_0) - E(Z_2) = 0 \]
\[ E(Y_2 | Y_3) = E(Y_2) = E(Z_0) - E(Z_2) = 0 \]

g. \( E(Y_2 | Y_4) \) is the MMSE estimate of \( Y_2 \), and \((Y_2, Y_4)\) are jointly Gaussian.

Because of that, it is just the linear estimate of \( Y_2 \) given \( Y_4 \).
\[ E(Y_2 | Y_4) = \frac{E(Y_2 Y_4)}{\sigma_Y^2} Y_4 = -\frac{1}{2} Y_4, \text{ where } E(Y_2 Y_4) = -1 \]

h. As in d., the probability distribution of \( Y_2 | Y_3, Y_4 \) is equal to that of \( Y_2 | Y_4 \).
\[ E(Y_2 | Y_3, Y_4) = E(Y_2 | Y_4) = -\frac{1}{2} Y_4 \]