1. Prove that the solution to the resource allocation problem below is max-min fair when $\alpha \to \infty$

$$
\max_x \sum_{i=1}^{N} U_i(x_i), \quad U_i(x_i) = \frac{x_i^{1-\alpha}}{1-\alpha}
$$

subject to

$$
\sum_{i=1}^{N} a_{ij} x_i \leq c_j, \quad j = 1, \cdots, m
$$

where $a_{ij} = 1$ if source $i$ routes over link $j$ and $a_{ij} = 0$ otherwise.

2. Show that in the sense of Lagrange Duality Theorem, the dual to the linear programming problem

$$
\min_x \quad b^T x
$$

subject to

$$
Ax \geq c, \\
x \geq 0,
$$

where $x$ and $b$ are $n$ dimensional vectors, $c$ is an $m$ dimensional vector, and $A$ is an $m \times n$ matrix, is another linear program:

$$
\max_\lambda \quad c^T \lambda
$$

subject to

$$
A^T \lambda \leq b, \\
\lambda \geq 0,
$$

3. Show that the sum of two convex mappings is also convex. Can you say this about the maximum of two convex mappings? How about the minimum?

4. A simple application of dual theory is quadratic programming:

$$
\min_x \quad \frac{1}{2} x^T Q x - b^T x
$$

subject to

$$
Ax \leq c.
$$

Assume $Q = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $A = [1, 1]$, and $c$ is a scalar. Provide a solution to this problem (as a function of $c$).
5. Consider three flows in the network. Each flow can adapt three rate control schemes: Inactive, TCP, and UDP. If a flow is inactive, it does not transmit any packets, hence does not affect other flows. If a flow is a TCP one, it accommodates the other flows by sharing it adaptively. However, if a flow is a UDP one, it will transmit aggressively by assuming a non-shared link. The payoff of each flow depends on the collective strategies of all flows in the following manner:

- The payoff for any flow that is inactive is 0.
- If two flows are inactive and one flow uses TCP or UDP it receives a payoff of 12.
- If one flow is inactive and two flows are TCP, the active flows each receive a payoff of 7.
- If one flow is inactive and two flows are UDP, the active flows each receive a payoff of 4.
- If one flow is inactive, one flow is TCP, and one UDP, the active flows receive payoff of 5 and 10, respectively.
- If one flow is TCP and the other two are UDP, the payoff of TCP flow is 1, while the payoff of each UDP flows is 3.
- If two flows are TCP and one UDP, the payoff of each of TCP flows is 3 while the UDP flow sees a payoff of 8.
- If all flows follow TCP, each receives a payoff of 6.
- If all flows follow UDP, each receives a payoff of 2.

Does this game have a Nash Equilibrium? If yes, how many? What is the worst case price of anarchy?