4.19 Prove that the convolution operation is associative for stable and single-sided sequences.

4.20 Develop closed-form expressions for the following convolution sums: (a) $\alpha^n \mu[n] \ast \mu[n]$, (b) $n\alpha^n \mu[n] \ast \mu[n]$.

4.21 Develop a general expression for the output $y[n]$ of an LTI discrete-time system in terms of its input $x[n]$ and the unit step response $s[n]$ of the system.

4.22 A periodic sequence $x[n]$ with a period $N$ is applied as an input to an LTI discrete-time system characterized by an impulse response $h[n]$ generating an output $y[n]$. Is $y[n]$ a periodic sequence? If it is, what is its period?

4.23 Let $y[n]$ be the sequence obtained by a linear convolution of two causal finite-length sequences $h[n]$ and $x[n]$. For each pair of $y[n]$ and $h[n]$ listed below, determine $x[n]$. The first sample in each sequence is at time index $n = 0$.

- (a) $\{y[n]\} = \{6, -7, -13, 24, 12, 33, -5, 14\}$, $\{h[n]\} = \{3, 4, -1, 2\}$,
- (b) $\{y[n]\} = \{-6, 17, 3, -38, 6, 41, -3, -20\}$, $\{h[n]\} = \{-2, 3, 1, 4\}$,
- (c) $\{y[n]\} = \{15, -4, -16, 26, 36, -12, -31, -12\}$, $\{h[n]\} = \{3, -2, 0, 5, 4\}$.

4.24 An LTI discrete-time system is characterized by a left-side impulse response given by $h[n] = \alpha^n \mu[-n - 1]$. Determine the range of the value of the constant $\alpha$ for which the system is BIBO stable.

4.25 Prove Eq. (4.28).

4.26 Prove Eq. (4.31).

4.27 Consider a cascade of two causal stable LTI systems characterized by impulse responses $\alpha^n \mu[n]$ and $\beta^n \mu[n]$, where $0 < \alpha < 1$ and $0 < \beta < 1$. Determine the expression for the impulse response $h[n]$ of the cascade.

4.28 Determine the impulse response $g[n]$ of the inverse system of the LTI discrete-time system of Example 4.16.

4.29 Determine the impulse response $g[n]$ characterizing the inverse system of the cascaded LTI discrete-time system of Problem 4.27.

4.30 Determine the expression for the impulse response of each of the LTI systems shown in Figure P4.1.

4.31 Determine the overall impulse response of the system of Figure P4.2, where the impulse responses of the component systems are: $h_1[n] = 2\delta[n - 2] + 3\delta[n - 1]$, $h_2[n] = \delta[n - 1] - 2\delta[n + 2]$, and $h_3[n] = 5\delta[n - 3] - 7\delta[n - 3] + 2\delta[n - 1] + \delta[n] - 3\delta[n + 1]$.
4.42 Determine the total solution for \( n \geq 0 \) of the difference equation
\[
y[n] - 0.16y[n - 1] = 5.88\mu[n],
\]
with the initial condition \( y[-1] = 5 \).

4.43 Determine the total solution for \( n \geq 0 \) of the difference equation
\[
y[n] - 0.7y[n - 1] - 0.02y[n - 2] = 3^n\mu[n],
\]
with the initial condition \( y[-1] = 3 \), and \( y[-2] = 0 \).

4.44 Determine the total solution for \( n \geq 0 \) of the difference equation
\[
y[n] - 0.3y[n - 1] - 0.04y[n - 2] = x[n] + 2x[n - 1],
\]
with the initial condition \( y[-1] = 3 \), and \( y[-2] = 0 \), when the forcing function is \( x[n] = 3^n\mu[n] \).

4.45 Determine the impulse response \( h[n] \) of the LTI system of Problem 4.42.

4.46 Determine the impulse response \( h[n] \) of the LTI system of Problem 4.44.

4.47 Determine the step response of an LTI discrete-time system characterized by an impulse response \( (-\alpha)^n\mu[n], 0 < \alpha < 1 \).

4.48 Show that the sum \( \sum_{n=0}^{\infty} |n^k(\lambda_1)^n| \) converges if \( |\lambda_1| < 1 \).

4.49 Let a causal IIR digital filter be described by the difference equation of Eq. (4.32) where \( y[n] \) and \( x[n] \) are the output and the input sequences, respectively. If \( h[n] \) denotes its impulse response, show that
\[
p_k = \sum_{n=0}^{k} h[n]d_{k-n}, \quad k = 0, 1, \ldots, M.
\]
From the above result, show that \( p_n = h[n] \oplus d_n \).

4.50 Consider a causal FIR filter of length \( L+1 \) with an impulse response given by \( \{g[n]\}, n = 0, 1, \ldots, L \). If the difference equation representation of the form of Eq. (4.32) where \( M + N = L \) of a causal finite-dimensional IIR digital filter with an impulse response \( \{h[n]\} \) such that \( h[n] = g[n] \) for \( n = 0, 1, \ldots, L \).

4.51 Compute the output of the accumulator of Eq. (4.2) for an input \( x[n] = n\mu[n] \) and the following initializations: (a) \( y[-1] = 1 \), (b) \( y[-1] = -1 \), and (c) \( y[-1] = 0 \).

4.52 In the rectangular method of numerical integration, the integral on the right-hand side of Eq. (4.61) is expressed as
\[
\int_{(n-1)T}^{nT} x(\tau)d\tau = T \cdot x ((n-1)T).
\]
Develop the difference equation representation of the rectangular method of numerical integration.

4.53 Develop a recursive implementation of the time-varying linear discrete-time system characterized by
\[
y[n] = \begin{cases} 
\frac{1}{n} \sum_{\ell=1}^{n} x[\ell], & n > 0, \\
0, & n \leq 0.
\end{cases}
\]
4.32 Is the cascade connection of two stable LTI systems also stable? Justify your answer.

4.33 Prove that the cascade connection of two passive (lossless) LTI systems is also passive (lossless).

4.34 Is the parallel connection of two stable LTI systems also stable? Justify your answer.

4.35 Is the parallel connection of two passive (lossless) LTI systems also passive (lossless)? Justify your answer.

4.36 Consider the causal LTI system described by the difference equation
\[ y[n] = p_0 x[n] + p_1 x[n - 1] - d_1 y[n - 1], \]
where \( x[n] \) and \( y[n] \) denote, respectively, its input and output. Determine the difference equation representation of its inverse system.

4.37 Derive Eq. (4.53) by induction by first evaluating Eq. (4.50) for \( n = 0, 1, 2, \ldots \), and then solving for \( h[0], h[1], h[2], \ldots \), etc.

4.38 Show by back substitution that the set of equations given by Eq. (4.55) reduces to Eq. (4.33).

4.39 The sequence of Fibonacci numbers \( f[n] \) is a causal sequence defined by
\[ f[n] = f[n - 1] + f[n - 2], \quad n \geq 2, \]
with \( f[0] = 0 \) and \( f[1] = 1 \).

(a) Develop an exact formula to calculate \( f[n] \) directly for any \( n \).

(b) Show that \( f[n] \) is the impulse response of a causal LTI system described by the difference equation [Joh89]
\[ y[n] = y[n - 1] + y[n - 2] + x[n - 1]. \]

4.40 Consider a first-order complex digital filter characterized by a difference equation
\[ y[n] = \alpha y[n - 1] + x[n], \]
where \( x[n] \) is the real input sequence, \( y[n] = y_{re}[n] + jy_{im}[n] \) is the complex output sequence with \( y_{re}[n] \) and \( y_{im}[n] \) denoting its real and imaginary parts, and \( \alpha = a + jb \) is a complex constant. Develop an equivalent two-output, single-input real difference equation representation of the above complex digital filter. Show that the single-input, single-output digital filter relating \( y_{re}[n] \) to \( x[n] \) is described by a second-order difference equation.

4.41 Let \( h[n], h[n + 1], \) and \( h[n + 2] \) denote the three consecutive impulse response samples of the first-order causal LTI system of Problem 4.36. Show that the coefficients of the difference equation, \( p_0, p_1, \) and \( d_1 \), characterizing this system can be uniquely determined from the above three impulse response samples.