Exercise Set #6

1. **Linear estimator.** Consider a channel with the observation \( Y = XZ \), where the signal \( X \) and the noise \( Z \) are uncorrelated Gaussian random variables. Let \( E[X] = 1 \), \( E[Z] = 2 \), \( \sigma_X^2 = 5 \), and \( \sigma_Z^2 = 8 \).

   (a) Find the MMSE linear estimate of \( X \) given \( Y \).

   (b) Suppose your friend from Caltech tells you that he was able to derive an estimator with a lower MSE. Your friend from MIT disagrees, saying that this is not possible because the signal and the noise are Gaussian, and hence the MMSE linear estimator will also be the MMSE estimator. Could your MIT friend be wrong?

2. **Additive-noise channel with path gain.** Consider the additive noise channel shown in the figure below, where \( X \) and \( Z \) are zero mean and uncorrelated, and \( a \) and \( b \) are constants.

   \[
   X \xrightarrow{a} Z \xrightarrow{b} Y = b(aX + Z)
   \]

   Find the MMSE linear estimate of \( X \) given \( Y \) and its MSE in terms only of \( \sigma_X \), \( \sigma_Z \), \( a \), and \( b \).

3. **Image processing.** A pixel signal \( X \sim U[-k, k] \) is digitized to obtain

   \[
   \hat{X} = i + \frac{1}{2}, \text{ if } i < X \leq i + 1, \ i = -k, -k+1, \ldots, k-2, k-1.
   \]

   To improve the the visual appearance, the digitized value \( \hat{X} \) is dithered by adding an independent noise \( Z \) with mean \( E[Z] = 0 \) and variance \( \text{Var}(Z) = N \) to obtain \( Y = \hat{X} + Z \).

   (a) Find the correlation of \( X \) and \( Y \).

   (b) Find the MMSE linear estimate of \( X \) given \( Y \). Your answer should be in terms only of \( k \), \( N \), and \( Y \).

4. **Noise cancellation.** A classical problem in statistical signal processing involves estimating a weak signal (e.g., the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations; one with the weak signal present and one without (by placing one microphone on the mother’s belly and another close to her heart). The observations can then be combined to estimate the weak signal by “canceling out” the interference. The following is a simple version of this application.

   Let the weak signal \( X \) be a random variable with mean \( \mu \) and variance \( P \), and the observations be \( Y_1 = X + Z_1 \) (\( Z_1 \) being the strong interference), and \( Y_2 = Z_1 + Z_2 \) (\( Z_2 \) is a measurement
noise), where $Z_1$ and $Z_2$ are zero mean with variances $N_1$ and $N_2$, respectively. Assume that $X$, $Z_1$ and $Z_2$ are uncorrelated. Find the MMSE linear estimate of $X$ given $Y_1$ and $Y_2$ and its MSE. Interpret the results.

5. **Nonlinear estimator.** Consider a channel with the observation $Y = XZ$, where the signal $X$ and the noise $Z$ are uncorrelated Gaussian random variables. Let $E[X] = 1$, $E[Z] = 2$, $\sigma^2_X = 5$, and $\sigma^2_Z = 8$.

   (a) Using the fact that $E(W^3) = \mu + 3\mu \sigma^2$ and $E(W^4) = \mu^4 + 6\mu^2 \sigma^2 + 3\sigma^4$ for $W \sim N(\mu, \sigma^2)$, find the mean and covariance matrix of $[X Y Y^2]^T$.

   (b) Find the MMSE linear estimate of $X$ given $Y$ and the corresponding MSE.

   (c) Find the MMSE linear estimate of $X$ given $Y^2$ and the corresponding MSE.

   (d) Find the MMSE linear estimate of $X$ given $Y$ and $Y^2$ and the corresponding MSE.

   (e) Compare your answers in parts (b) through (d). Is the MMSE estimate of $X$ given $Y$ (namely, $E(X|Y)$) linear? This would answer Problem 1 in Homework Set #6.

6. **Minimum waiting time.** Let $X_1, X_2, \ldots$ be i.i.d. exponentially distributed random variables with parameter $\lambda$, i.e. $f_{X_i}(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.

   (a) Does $Y_n = \min\{X_1, X_2, \ldots, X_n\}$ converge in probability as $n$ approaches infinity?

   (b) If it converges what is the limit?

   (c) What about $Z_n = nY_n$?

7. **Roundoff errors.** The sum of a list of 100 real numbers is to be computed. Suppose that these numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Use the central limit theorem to estimate the probability that the total error in the sum of the 100 numbers exceeds 6.

8. **Polya urn.** An urn initially has one red ball and one white ball. Let $X_1$ denote the name of the first ball drawn from the urn. Replace that ball and one like it. Let $X_2$ denote the name of the next ball drawn. Replace it and one like it. Continue, drawing and replacing.

   (a) Argue that the probability of drawing $k$ reds followed by $n - k$ whites is

   \[\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{k}{k+1} \cdot \frac{1}{k+2} \cdots \frac{1}{n+1} = \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1} \binom{n}{k}.\]

   (b) Let $P_n$ be the proportion of red balls in the urn after the $n^{th}$ drawing. Argue that $\Pr\{P_n = \frac{k}{n+2}\} = \frac{1}{n+1}$, for $k = 1, 2, \ldots, n+1$. Thus all proportions are equally probable. This shows that $P_n$ tends to a uniformly distributed random variable in distribution, i.e.,

   \[\lim_{n \to \infty} \Pr\{P_n \leq t\} \longrightarrow t, \quad 0 \leq t \leq 1.\]

   (c) What can you say about the behavior of the proportion $P_n$ if you started initially with one red ball in the urn and two white balls? Specifically, what is the limiting distribution of $P_n$? Can you show $\Pr\{P_n = \frac{k}{n+3}\} = \frac{k}{n+2}$, for $k = 1, 2, \ldots, n+1$?