Exercise Set #7

1. **Symmetric random walk.** Let $X_n$ be a random walk defined by

\[
X_0 = 0, \\
X_n = \sum_{i=1}^{n} Z_i,
\]

where $Z_1, Z_2, \ldots$ are i.i.d. with $P\{Z_1 = -1\} = P\{Z_1 = 1\} = \frac{1}{2}$.

(a) Find $P\{X_{10} = 10\}$.
(b) Approximate $P\{-10 \leq X_{100} \leq 10\}$ using the central limit theorem.
(c) Find $P\{X_n = k\}$.

2. **Absolute-value random walk.** Consider the symmetric random walk $X_n$ in the previous problem. Define the absolute value random process $Y_n = |X_n|$.

(a) Find $P\{Y_n = k\}$.
(b) Find $P\{\max_{1 \leq i < 20} Y_i = 10 | Y_{20} = 0\}$.

3. **A random process.** Let $X_n = Z_{n-1} + Z_n$ for $n \geq 1$, where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0,1)$.

(a) Find the mean and autocorrelation functions of $\{X_n\}$.
(b) Is $\{X_n\}$ wide-sense stationary? Justify your answer.
(c) Is $\{X_n\}$ Gaussian? Justify your answer.
(d) Is $\{X_n\}$ strict-sense stationary? Justify your answer.
(e) Find $E(X_3|X_1, X_2)$.
(f) Find $E(X_3|X_2)$.
(g) Is $\{X_n\}$ Markov? Justify your answer.
(h) Is $\{X_n\}$ independent increment? Justify your answer.

4. **Moving average process.** Let $X_n = \frac{1}{2}Z_{n-1} + Z_n$ for $n \geq 1$, where $Z_0, Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0,1)$. Find the mean and autocorrelation function of $X_n$.

5. **Autoregressive process.** Let $X_0 = 0$ and $X_n = \frac{1}{2}X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\sim N(0,1)$. Find the mean and autocorrelation function of $X_n$. 


6. Random binary waveform. In a digital communication channel the symbol “1” is represented by the fixed duration rectangular pulse

\[ g(t) = \begin{cases} 
1 & \text{for } 0 \leq t < 1 \\
0 & \text{otherwise},
\end{cases} \]

and the symbol “0” is represented by \(-g(t)\). The data transmitted over the channel is represented by the random process

\[ X(t) = \sum_{k=0}^{\infty} A_k g(t - k), \quad t \geq 0, \]

where \(A_0, A_1, \ldots\) are i.i.d random variables with

\[ A_i = \begin{cases} 
+1 & \text{w.p. } \frac{1}{2} \\
-1 & \text{w.p. } \frac{1}{2}.
\end{cases} \]

(a) Find its first and second order pmfs.
(b) Find the mean and the autocorrelation function of the process \(X(t)\).

7. QAM random process. Consider the random process

\[ X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t, \quad -\infty < t < \infty, \]

where \(Z_1\) and \(Z_2\) are i.i.d. discrete random variables such that \(p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}\).

(a) Is \(X(t)\) wide-sense stationary? Justify your answer.
(b) Is \(X(t)\) strict-sense stationary? Justify your answer.

8. Mixture of two WSS processes. Let \(X(t)\) and \(Y(t)\) be two zero-mean WSS processes with autocorrelation functions \(R_X(\tau)\) and \(R_Y(\tau)\), respectively. Define the process

\[ Z(t) = \begin{cases} 
X(t), & \text{with probability } \frac{1}{2} \\
Y(t), & \text{with probability } \frac{1}{2}.
\end{cases} \]

Find the mean and autocorrelation functions for \(Z(t)\). Is \(Z(t)\) a WSS process? Justify your answer.

9. Stationary Gauss-Markov process. Let

\[ X_0 \sim N(0, a) \]

\[ X_n = \frac{1}{2} X_{n-1} + Z_n, \quad n \geq 1, \]

where \(Z_1, Z_2, Z_3, \ldots\) are i.i.d. \(N(0,1)\) independent of \(X_0\).

(a) Find \(a\) such that \(X_n\) is stationary. Find the mean and autocorrelation functions of \(X_n\).
(b) (Difficult.) Consider the sample mean \(S_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad n \geq 1\). Show that \(S_n\) converges to the process mean in probability even though the sequence \(X_n\) is not i.i.d. (A stationary process for which the sample mean converges to the process mean is called mean ergodic.)