Kolmogorov - Axioms On Probability

\[ P: \mathbb{F} \rightarrow \mathbb{R} \] satisfies:

1. \( P(A) \geq 0 \) for every \( A \in \mathbb{F} \)

2. \( P(\emptyset) = 1 \)

3. (Countable Additivity) If \( A_1, A_2, \ldots \) are disjoint (i.e., \( A_i \cap A_j = \emptyset \) for all \( i \neq j \))

\[ P(\bigcup_i A_i) = \sum_i P(A_i) \]

Law Of Total Probability

1. \( P(A^c) = 1 - P(A) \)

Examples:

1. Two Coin Flips:
   \[ \mathcal{F} = \{ H, T \} \]
   \[ \mathbb{F} = \{ \emptyset, \{ H \}, \{ T \}, \{ H, T \} \} \]
   \[ P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2} \]

2. Two Coin Flips:
   \[ \mathcal{F} = \{ \emptyset, \{ H \}, \{ T \}, \{ H, T \} \} \]
   \[ \mathbb{F} = \{ \emptyset, \{ H \}, \{ T \}, \{ H, T \} \} \]
   \[ P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2} \]

3. Four Coin Flips 
   \[ \mathcal{F} = \{ \emptyset, \{ H \}, \{ T \}, \{ H, T \} \} \]
   \[ \mathbb{F} = \{ \emptyset, \{ H \}, \{ T \}, \{ H, T \} \} \]
   \[ P(H) = \frac{1}{2}, \quad P(T) = \frac{1}{2} \]

4. Roll A Fair Die
   \[ \mathcal{F} = \{ 1, 2, 3, 4, 5, 6 \} \]
   \[ \mathbb{F} = \{ \emptyset, \{ 1 \}, \{ 2 \}, \{ 3 \}, \{ 4 \}, \{ 5 \}, \{ 6 \} \} \]

5. Roll A Fair Die
\[ C = \{ 1, 2, 3, 4, 5, 6 \} \]
\[ P(C) = 2^6 \]
\[ P(\emptyset) = 0, \quad P(C) = P(\emptyset) + P(C) + \ldots + P(\emptyset) + \ldots \]
\[ P(\{1, 2\}) = \frac{P(\{1, 2\})}{\ldots + P(\{1, 2\})} = \frac{1}{2} \]
\[ \frac{P(\{1, 2\})}{\ldots + \frac{P(\{5, 6\})}{2}} \]
\[ P(\{5\}) = 1 \]

**Exercise**

**Known From Generation**

1. \( \mathcal{C} = \{0, 1\} \)
2. \( P = \{a, b\} \)
3. \( P(a) = a, \quad P(\{a\}) \quad \forall a \in \{0, 1\} \)

If \( \mathcal{C} \) is Continuous (e.g., is the Interval on the Real Line) \( \forall \mathcal{C} \) Rules, Then \( P(a) \) \( \forall a \in \mathcal{C} \) Determines \( \forall A \) for Every \( A \in \mathcal{C} \)

For Instance

\[ P(\{a\}) = \int_{a}^{b} P \left( \frac{\{a\}}{\mathcal{C}} \right) \]

A Similar Conclusion Can Be Made w/ \( P(\{a\}) \) \( \forall a \in \mathcal{C} \)

For Instance

\[ P(\{a\}) = \int_{a}^{b} P \left( \frac{\{a\}}{\mathcal{C}} \right) \]

**Conditional Probability**

If \( B \) be an Event \( \forall P(B) \neq 0 \), Then \( \frac{P(\{A\})}{P(B)} \)

\[ = \frac{\sum P(\{A\} \cap B)}{P(B)} \]

Note that \( P(\{A\}) \) is a probability measure over \( \mathcal{C} \) to meet the 3 Axioms (1) \( P(\{A\}) \neq 0 \) \( \forall A \)

2. \( P(\{A\} \cap B) = P(\{A\}) \cap P(B) \)

\[ = \frac{\sum P(\{A\} \cap B)}{P(B)} = \frac{P(\{A\} \cap B)}{P(B)} \]

**Chain Rule** \( \text{- True Even When} \ P(B) \) or \( P(A) = 0 \)

\[ P(\{A\} \cap B) = P(\{A\}) \cap P(B) \]

\( \text{- Can Be Generalized To More Events} \)

\[ P(\{A, A, A\}) = P(\{A\}) \cap P(\{A\}) \cap P(\{A\}) \]

\[ = P(\{A\} \neq 0, \text{Then} \ P(A) \neq 0, \text{Then} \ P(A) \neq 0, \text{Then} \ P(A) \neq 0, \text{Then} \ P(A) \neq 0, \text{Then} \ P(A) \neq 0, \text{Then} \]

**Axioms**

1. \( \text{Axiom} \text{ of} \text{Probability} \)

\[ P(\{A\}) = P(\{A\}) \cap P(\{A\}) \]

\[ = \frac{P(\{A\} \cap B)}{P(B)} = \frac{P(\{A\} \cap B)}{P(B)} \]

\[ = \frac{P(\{A\} \cap B)}{P(B)} = \frac{P(\{A\} \cap B)}{P(B)} \]

**Example**

1. **Axiom of Probability (Cont.)**

\[ P(\{A\}) = \frac{1}{2}, \quad \frac{P(\{A\})}{n} = \frac{1}{2}, \quad \frac{P(\{A\})}{n} = \frac{1}{2} \]

\[ \frac{P(\{A\})}{n} = \frac{1}{2}, \quad \frac{P(\{A\})}{n} = \frac{1}{2} \]
\[ A = \{(0, 0), (0, 1), (1, 0), (1, 1)\} \]

Let \( A = \{0, 1\} \) be the set of states.
Let \( B = \{0, 1\} \) be the set of actions.

\[ \text{Find } P(A) = 0.2 \]
\[ P(A) \cap \{0, 1\} = 0.7 \cdot 0.9 + 0.8 \cdot 0.25 \]
\[ P(A) = 0.9 \]
\[ P(X) = \frac{P(A) \cdot P(Y)}{P(B)} \]

**Ex 2: Finite State Machine (3 States)**

\[ P(A) = \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.2 & 0.3 & 0.4 \\ 0.1 & 0.3 & 0.6 \end{pmatrix} \]

\[ \text{Let } A = \{\text{Initial State}, \text{Next State}\} \]

\[ A = \{X, Y, Z\} \]

\[ \text{Find } P(A) = 0.5 \]
\[ P(A) = P(\text{Initial State}, \text{Next State}) = P(X, Y) + P(Y, X) \]
\[ P(A) = 0.3 \]
\[ P(Y) = \frac{P(A) \cdot P(Y)}{P(B)} = 0.5 \cdot 0.7 + 0.8 \cdot 0.25 = 0.7 + 0.3 \cdot 0.5 \]