**Mean & Variance**

- **The First Moment (or Mean) of X** is: \( E[X] = \int_{-\infty}^{\infty} x f(x) \, dx \).

  **Proof**: If \( X \geq 0 \) then \( E[X] = \int_{0}^{\infty} x f(x) \, dx \).

**Second Moment (or Average Power) of X** is: \( E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx \)

**The Variance (or Mean Squared Error) of X** is: \( \text{Var}(X) = E[(X - E(X))^2] = E[X^2] - 2E[X \cdot E(X)] + [E[X]]^2 \)

**Show**: If \( \text{Var}(X) \geq 0 \) then \( E[X^2] \geq [E[X]]^2 \).
- Standard Deviation (Sometimes Useful Even When Units Are X)
  \[ \sigma_X = \sqrt{\text{Var}(X)} \quad \text{i.e.} \quad \text{Var}(X) = \sigma_X^2 \]

Mean & Var of Famous Distributions:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>(E[X])</th>
<th>(\text{Var}(X))</th>
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<tbody>
<tr>
<td>Bernoulli((p))</td>
<td>(p)</td>
<td>(p(1-p))</td>
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<tr>
<td>Geometric</td>
<td>(\frac{1}{p})</td>
<td>(\frac{1}{p^2})</td>
</tr>
<tr>
<td>Binomial((n, p))</td>
<td>(np)</td>
<td>(np(1-p))</td>
</tr>
<tr>
<td>Poisson((\lambda))</td>
<td>(\lambda)</td>
<td>(\lambda)</td>
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<tr>
<td>Uniform((a, b))</td>
<td>(\frac{a+b}{2})</td>
<td>(\frac{(b-a)^2}{12})</td>
</tr>
<tr>
<td>Exponential((\lambda))</td>
<td>(\frac{1}{\lambda})</td>
<td>(\lambda^2)</td>
</tr>
<tr>
<td>(N(\mu, \sigma^2))</td>
<td>(\mu)</td>
<td>(\sigma^2)</td>
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</tbody>
</table>

Note \(E[X]\) can be \(\infty\) or may not exist

Ex: 1. \(f(x) = \frac{1}{x^2}, x \geq 2\)
\[
E[X] = \int_2^{\infty} x f(x) \, dx = \int_2^{\infty} \frac{x}{x^2} \, dx + \int_0^2 \frac{x}{x^2} \, dx = \frac{1}{2} + \infty = ???
\]

Thomas Hardy, Littlewood, Book on Inequalities, Cambridge Univ Press

Inequalities

1. **Markov Inequality**
   
   If \(X \geq 0\) with \(E[X] = \mu < \infty\), then \(P(X \geq \alpha \mu) \leq \frac{1}{\alpha^2}\), \(\forall \alpha > 1\)

   Proof: \(\mu = E[X] = \int_{-\infty}^{\infty} X \cdot f(x) \, dx \geq \frac{1}{\alpha^2} \cdot \mu \cdot f(\alpha \mu) \cdot \int_{-\infty}^{\alpha \mu} \, dx \)

2. **Chebyshev Inequality**
   
   For any random variable \(X\) with \(\mu \leq \infty\) and \(\text{Var}(X) = \sigma_X^2 < \infty\), then \(P(|X - \mu| \geq \alpha \sigma_X) \leq \frac{1}{\alpha^2}\), \(\forall \alpha > 1\)

   Proof: \(P(|X - \mu| \geq \alpha \sigma_X) = P((X - \mu)^2 \geq \alpha^2 \sigma_X^2) = \frac{1}{\alpha^2} P(X^2) \geq \alpha^2 \text{Var}(X) \leq \frac{1}{\alpha^2}\) by Markov
**3)** \[**Tensens Inequality**

A function \( g(x) \) is said to be convex if you draw a line segment between any two points, the line segment must lie above \( g(x) \).

\[ g(x) \leq \frac{g(a) + g(b)}{2} \quad \text{for all } x \in [a, b] \quad \text{for all } a, b \]

If the function has a 2nd derivative, then \( g(x) \) is convex if \( g''(x) \geq 0 \).

**Ex:**

\( g(x) = x^2 + 6 \)

\( g''(x) = 2 \)

If \( g(x) = x^2 \), \( x > 0 \)

**Def:** If \(-g(x)\) is convex, then \( g(x) \) is **concave**.

**Tensens Ineq:** If \( g(x) \) is convex, then \( E[g(x)] \geq g(E[x]) \)

If \( g(x) \) is concave, then \( E[g(x)] \leq g(E[x]) \)

**Applications:**

\( g(x) = x^2 \Rightarrow E[x^2] \geq (E[x])^2 \)

**Monotonicity of Means**

\( E[|X|^p] \geq (E[|X|])^p \) for \( p \geq 2 \)

**Assume** \( g(x) = x^p \) for \( x \geq 0 \) **Absolute Growth From Two Linear Convex**

\[ E[(X)^p] \geq (E[X])^p \]

\[ E[(|X|)^p] \geq (E[|X|])^p \]

Take \((*)^p\) On both Sides

**C** Let \( g(x) = \frac{1}{x} \) \( x > 0 \)

**By Tensens** \( E[(X)^p] \geq E[(X)]^p \) Let \((*) = x^p \) Then \( E[\frac{1}{x^p}] \geq E[\frac{1}{x}] \)
Expectation Involving 2 RVs

Let Expectation of \( Z \cdot g(X,Y) \) as \( f_{Z}(z) \). Let \( E[Z] = \int_{-\infty}^{\infty} f_{Z}(z) \, dz \). Also, \( E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) \, dx \, dy = E[Z|X,Y] \).

Ex. 1: Correlation \( E[X^2] \). We say that \( X \) and \( Y \) are orthogonal if \( E[X^2] = 0 \).


Covariance \( (X,Y) = \text{Var}(X) \)

We say \( X \) and \( Y \) are uncovariance if \( \text{Cov}(X,Y) = 0 \).

- \( X, Y \) are independent \( \implies X, Y \) are uncorrelated

Prove \( \int x f_{X}(x) f_{Y}(y) \, dx \, dy = \int x f_{X}(x) \, dx \int f_{Y}(y) \, dy \).

That is \( E[X|Y] = E[X]E[Y] \).

The converse is not true.

- \( X, Y \) are uncorrelated \( \not\implies X, Y \) are independent

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