1. Order statistics.
Let $X_1, X_2, X_3$ be independent and uniformly drawn from the interval $[0, 1]$. Let $Y_1$ be the smallest of $X_1, X_2, X_3$, let $Y_2$ be the median (second smallest) of $X_1, X_2, X_3$, and let $Y_3$ be the largest of $X_1, X_2, X_3$. For example, if $X_1 = .3, X_2 = .1, X_3 = .7$, then $Y_1 = .1, Y_2 = .3, Y_3 = .7$. The random variables $Y_1, Y_2, Y_3$ are called the order statistics of $X_1, X_2, X_3$.

(a) What is the probability $P\{X_1 \leq X_2 \leq X_3\}$?
(b) Find the pdf of $Y_1$.
(c) Find the pdf of $Y_3$.
(d) (Difficult.) Find the pdf of $Y_2$.
(Hint: $Y_2 \leq y$ if and only if at least two among $X_1, X_2, X_3$ are $\leq y$.)

Solution:

(a) By symmetry, $P\{X_i \leq X_j \leq X_k\}$ should be identical for all $i \neq j \neq k$. Since there are $3! = 6$ such $(i, j, k)$, the probability should be $1/6$.

(b) We have $P\{Y_1 > y\} = P\{X_1, X_2, X_3 > y\} = P\{X_1 > y\}P\{X_2 > y\}P\{X_3 > y\} = (1-y)^3$. Hence, $f_{Y_1}(y) = \frac{d}{dy}(1-(1-y)^3) = 3(1-y)^2$ for $0 \leq y \leq 1$.

(c) We can use the similar steps to part (b) to find $f_{Y_3}(y) = 3y^2$, $0 \leq y \leq 1$. Alternatively, we can see that by symmetry $1-Y_3$ and $Y_1$ should have the same pdf, which gives the same answer.

(d) The event $\{Y_2 \leq y\}$ can be expressed as the union of following mutually exclusive events

$$\{Y_2 \leq y\} = \{X_1, X_2 \leq y, X_3 > y\} \cup \{X_2, X_3 \leq y, X_1 > y\} \cup \{X_3, X_1 \leq y, X_2 > y\} \cup \{X_1, X_2, X_3 \leq y\}.$$ 

By symmetry $P\{X_1, X_2 \leq y, X_3 > y\} = P\{X_2, X_3 \leq y, X_1 > y\} = P\{X_3, X_1 \leq y, X_2 > y\} = y^2(1-y)$. Hence, $P\{Y_2 \leq y\} = 3y^2(1-y) + y^3 = 3y^2 - 2y^3$. By taking the derivative, we have $f_{Y_2}(y) = 6y(1-y)$ for $0 \leq y \leq 1$. This distribution is known as the Beta($2, 2$) distribution.
2. **Winner of a race.**

Two horses are racing on a track. Let $X$ and $Y$ be the finish times of horse 1 and horse 2, respectively. Suppose $X$ and $Y$ are independent and identically distributed $\text{Exp}(1)$ random variables, that is,

$$
P\{X > x, Y > y\} = e^{-x} e^{-y}
$$

for all $x, y \geq 0$. Let $W$ denote the index of the winning horse. Then $W = 1$ (i.e., horse 1 wins the race) if $X < Y$, and $W = 2$ if $X \geq Y$.

(a) Find $P\{W = 2\}$.

(b) Find $P\{W = 2 \mid Y = y\}$ for $y \geq 0$.

(c) Suppose we wish to guess which horse won the race based on the finish time of one horse only, say, $Y$. Find the optimal decision rule $D(y)$ that minimizes the probability of error $P\{W \neq D(Y)\}$.

(d) Find the minimum probability of error in part (c).

**Hint:** The following facts might be useful:

\[
\begin{align*}
\int_0^t e^{-x} \, dx &= 1 - e^{-t}, \\
\int_t^\infty e^{-x} \, dx &= e^{-t}, \\
\int_0^t e^{-2x} \, dx &= \frac{1}{2} (1 - e^{-2t}), \\
\int_t^\infty e^{-2x} \, dx &= \frac{1}{2} e^{-2t}, \\
e^{-\ln 2} &= \frac{1}{2}, \\
e^{-2\ln 2} &= \frac{1}{4}.
\end{align*}
\]

**Solution:**

(a) By symmetry, $P\{W = 1\} = P\{W = 2\} = 1/2$. Alternatively, we have

$$
P\{W = 2\} = P\{Y \leq X\} = \int_0^\infty \int_y^\infty f_X(x) f_Y(y) \, dx \, dy = \int_0^\infty e^{-y} f_Y(y) \, dy = \int_0^\infty e^{-2y} \, dy = \frac{1}{2}.
$$
(b) For \( y \geq 0 \), we have
\[
\mathbb{P}\{W = 2 \mid Y = y\} = \mathbb{P}\{Y \leq X \mid Y = y\} \\
= \mathbb{P}\{y \leq X \mid Y = y\} \\
= \mathbb{P}\{y \leq X\} \\
= \int_y^\infty f_X(x) \, dx \\
= \int_y^\infty e^{-x} \, dx \\
= e^{-y}.
\]

(c) The optimal decision rule that minimizes the probability of error \( \mathbb{P}\{W \neq D(Y)\} \) is the MAP decoding rule:
\[
D(y) = \begin{cases} 
1 & \text{if } \mathbb{P}\{W = 1 \mid Y = y\} \geq \mathbb{P}\{W = 2 \mid Y = y\}, \\
2 & \text{otherwise}.
\end{cases}
\]

So we first find the \textit{a posteriori} probability. For \( y \geq 0 \),
\[
\mathbb{P}\{W = 2 \mid Y = y\} = e^{-y}, \\
\mathbb{P}\{W = 1 \mid Y = y\} = 1 - \mathbb{P}\{W = 2 \mid Y = y\} = 1 - e^{-y}.
\]

By letting \( \mathbb{P}\{W = 1 \mid Y = y\} = \mathbb{P}\{W = 2 \mid Y = y\} \), we get the threshold \( 1 - e^y = e^y \), or equivalently, \( y = \ln 2 \). Therefore,
\[
D(y) = \begin{cases} 
1, & \text{if } y \geq \ln 2, \\
2, & \text{if } y < \ln 2.
\end{cases}
\]

(d) We have
\[
\mathbb{P}\{W \neq D(Y)\} = \mathbb{P}\{D(Y) = 2, W = 1\} + \mathbb{P}\{D(Y) = 1, W = 2\} \\
= \mathbb{P}\{X < Y < \ln 2\} + \mathbb{P}\{\ln 2 \leq Y \leq X\} \\
= \int_0^{\ln 2} \int_0^y f_X(x) f_Y(y) \, dx \, dy + \int_y^\infty \int_y^\infty f_X(x) f_Y(y) \, dx \, dy \\
= \int_0^{\ln 2} \int_0^y e^{-x} e^{-y} \, dx \, dy + \int_y^\infty \int_y^\infty e^{-x} e^{-y} \, dx \, dy \\
= \int_0^{\ln 2} e^{-y} (1 - e^{-y}) \, dy + \int_{\ln 2}^\infty e^{-2y} \, dy \\
= \int_0^{\ln 2} e^{-y} \, dy - \int_0^{\ln 2} e^{-2y} \, dy + \int_{\ln 2}^\infty e^{-2y} \, dy \\
= \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{1}{4} \right) + \frac{1}{2} \cdot \frac{1}{4} \\
= \frac{1}{4}.
\]
3. **Estimation.** Let $X$ and $Y$ be independent and identically distributed random variables, $X, Y \sim \text{Unif}[-\frac{1}{2}, \frac{1}{2}]$ and let $Z = X + Y^2$.

(a) Find the conditional density $f_{Z|X}(z|y)$.
(b) Find the MMSE estimate of $Z$ given $Y$.
(c) Find the expected estimation error of the MMSE estimate in part (b).

**Solution:**

(a) We have

$$F_{Z|Y}(z|y) = P\{Z \leq z \mid Y = y\} = P\{X \leq z - y^2 \mid Y = y\} = P\{X \leq z - y^2\}$$

$$= \begin{cases} 0 & \text{if } z - y^2 < -\frac{1}{2}, \\ z - y^2 + \frac{1}{2} & \text{if } -\frac{1}{2} \leq z - y^2 \leq \frac{1}{2}, \\ 1 & \text{if } z - y^2 > \frac{1}{2}. \end{cases}$$

Therefore,

$$f_{Z|Y}(z|y) = \begin{cases} 1 & \text{if } y^2 - \frac{1}{2} \leq z \leq y^2 + \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$$

(b) We have

$$E(Z \mid Y = y) = \int_{-\infty}^{\infty} z f_{Z|Y}(z|y)dz = \int_{y^2 - \frac{1}{2}}^{y^2 + \frac{1}{2}} z dz = \frac{1}{2}((y^2 + \frac{1}{2})^2 - (y^2 - \frac{1}{2})^2) = y^2,$$

and hence

$$E(Z \mid Y) = Y^2.$$

(c) $\text{MSE} = E(\text{Var}(Z|Y)) = E(E(Z^2|Y) - (E(Z|Y))^2)$.

Using the conditional density of part (a) we have

$$E(Z^2|Y = y) = \int_{-\infty}^{\infty} z^2 f_{Z|Y}(z|y)dz = \int_{y^2 - \frac{1}{2}}^{y^2 + \frac{1}{2}} z^2 dz = \frac{1}{12} + y^4.$$ 

Therefore,

$$\text{MSE} = E\left(\frac{1}{12} + Y^4 - (Y^2)^2\right) = \frac{1}{12}.$$
4. Z channel.

Suppose that the signal $X$ is drawn as

$$X = \begin{cases} 
1 & \text{with probability } \frac{1}{2}, \\
0 & \text{with probability } \frac{1}{2}, 
\end{cases}$$

and the conditional pmf $p_{Y|X}(y|x)$ of $Y$ given $X$ is specified by

$$p_{Y|X}(1|1) = 1, \\
p_{Y|X}(1|0) = \frac{1}{2}.$$

(a) Find $p_{Y|X}(0|1)$ and $p_{Y|X}(0|0)$.

(b) Find the conditional pmf $p_{X|Y}(x|y)$ of $X$ given $Y$ (i.e., find $p_{X|Y}(1|1), p_{X|Y}(1|0), p_{X|Y}(0|1),$ and $p_{X|Y}(0|0))$.

(c) Find the optimal decoder $d(y)$ that minimizes the probability of error $P\{X \neq d(Y)\}$.

(d) Find the associated probability of error.

Solution:

(a) It is easy to see that $p_{Y|X}(0|1) = 1 - p_{Y|X}(1|1) = 0$ and $p_{Y|X}(0|0) - p_{Y|X}(1|0) = 1/2$.

(b) By the Bayes rule,

$$p_{X|Y}(1|1) = \frac{p_X(1)p_{Y|X}(1|1)}{p_Y(1)} = \frac{p_X(1)p_{Y|X}(1|1)}{p_X(1)p_{Y|X}(1|1) + p_X(0)p_{Y|X}(1|0)} = \frac{2}{3}.$$

And similarly, $p_{X|Y}(1|0) = 0$, $p_{X|Y}(0|1) = 1/3$, and $p_{X|Y}(0|0) = 1$.

(c) Since $p_{X|Y}(1|1) > p_{X|Y}(0|1)$ and $p_{X|Y}(0|0) > p_{X|Y}(1|0)$, the optimal decoder $d(y)$ is given by $d(1) = 1$ and $d(0) = 0$.

(d) We have

$$P\{X \neq d(Y)\} = P\{X \neq d(Y)|X = 1\}p_X(1) + P\{X \neq d(Y)|X = 0\}p_X(0)$$

$$= \frac{1}{2}P\{Y = 0|X = 1\} + \frac{1}{2}P\{Y = 1|X = 0\}$$

$$= \frac{1}{4}.$$