1. *Nonlinear processing.* Let $X \sim \text{Unif}[-1, 1]$. Define the random variable

$$Y = \begin{cases} 
X^2 + 1, & \text{if } |X| \geq 0.5 \\
0, & \text{otherwise.}
\end{cases}$$

Find and sketch the cdf of $Y$.

**Solution:** First we note that $Y \geq 0$ and thus for $y < 0$, $F_Y(y) = P(Y \leq y) = 0$.

It can be easily shown (Hint: see problem 11) that $|X| \sim \text{Unif}[0,1]$.

We have $P(Y = 0) = P(|X| < 0.5) = 1/2$.

For $y > 0$, we have

$$F_Y(y) = P(Y \leq y) = P(Y = 0) + P(0 < Y \leq y) = 1/2 + P(|X| \geq 0.5, X^2 + 1 \leq y)$$

$$= 1/2 + P(|X| \in (0.5, \sqrt{y-1}])$$

$$= \begin{cases} 
1/2, & \sqrt{y-1} \leq 0.5 \\
1/2 + \sqrt{y-1} - 1/2, & \text{otherwise}
\end{cases}$$

Collecting the results, we have

$$F_Y(y) = \begin{cases} 
0, & y < 0 \\
1/2, & 0 \leq y < 1.25 \\
\sqrt{y-1}, & 1.25 \leq y < 2 \\
1, & y \geq 2.
\end{cases}$$

The cdf is plotted in Figure 1.

2. *Geometric with conditions.* Let $X$ be a geometric random variable with pmf

$$p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots.$$
Find and plot the conditional pmf $p_X(k|A) = P\{X = k|X \in A\}$ if:

(a) $A = \{X > m\}$ where $m$ is a positive integer.
(b) $A = \{X < m\}$.
(c) $A = \{X$ is an even number$\}$.

Comment on the shape of the conditional pmf of part (a).

**Solution:**

(a) We have

$$P(A) = \sum_{n=m+1}^{\infty} p(1-p)^{n-1}$$

$$= \sum_{n=0}^{\infty} p(1-p)^{n+m}$$

$$= p(1-p)^m \sum_{n=0}^{\infty} (1-p)^n$$

$$= (1-p)^m.$$

For $k \leq m$, $p_X(k|A) = 0$. For $k > m$,

$$p_X(k|A) = P\{X = k|X > m\}$$

$$= \frac{P\{X = k\}}{P\{X > m\}}$$

$$= \frac{p(1-p)^{k-1}}{(1-p)^m}$$

$$= p(1-p)^{k-m-1}.$$
(b) We have

\[
P(A) = \sum_{n=0}^{m-2} p(1-p)^n \\
= p \frac{1 - (1-p)^{m-1}}{1 - (1-p)} \\
= 1 - (1-p)^{m-1}.
\]

For \( k \geq m \) or \( k \leq 0 \), \( p_X(k|A) = 0 \). For \( 0 < jk < m \),

\[
p_X(k|A) = P\{X = k|X < m\} \\
= \frac{P\{X = k\}}{P\{X < m\}} \\
= \frac{p(1-p)^{k-1}}{1 - (1-p)^{m-1}}.
\]

(c) We have

\[
P(A) = \sum_{n \text{ even}} p(1-p)^{n-1} \\
= \sum_{n' = 0}^{\infty} p(1-p)((1-p)^2)^{n'} \\
= \frac{p(1-p)}{1 - (1-p)^2} \\
= \frac{1-p}{2-p}.
\]

For \( k \) odd, \( P_X(k|A) = 0 \). For \( k \) even,

\[
p_X(k|A) = P\{X = k|X \text{ is even}\} \\
= \frac{P\{X = k\}}{P\{X \text{ is even}\}} \\
= \frac{p(1-p)^{k-1}}{P(A)} \\
= p(2-p)(1-p)^{k-2}.
\]

Plots are shown in Figure 2. The shape of the conditional pmf in part (a) shows that the geometric random variable is memoryless:

\[
p_X(x|X > k) = p_X(x-k), \quad \text{for } x \geq k.
\]

Note that in all three parts \( p_X(x) \) is defined for all \( x \). This is required.

3. Let \( A \) be a nonzero probability event. Show that
(a) \( P(A) = P(A|X \leq x)F_X(x) + P(A|X > x)(1 - F_X(x)) \).

(b) \( F_X(x|A) = \frac{P(A|X \leq x)}{P(A)}F_X(x) \).

Solution:

(a) By law of total probability we have

\[
P(A) = P(A|X \leq x)P(X \leq x) + P(A|X > x)P(X > x)
= P(A|X \leq x)F_X(x) + P(A|X > x)(1 - F_X(x)).
\]

(b) \( F_X(x|A) = P(X \leq x|A) = \frac{P(A|X \leq x)}{P(A)}P(X \leq x) = \frac{P(A|X \leq x)}{P(A)}F_X(x). \)
4. **Joint cdf or not.** Consider the function

$$G(x, y) = \begin{cases} 1 & \text{if } x + y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Can $G$ be a joint cdf for a pair of random variables? Justify your answer.

**Solution:** No. Note that for every $x$,

$$\lim_{y \to \infty} G(x, y) = 1.$$  

But for any genuine marginal cdf,

$$\lim_{x \to -\infty} F_X(x) = 0 \neq 1.$$  

Therefore $G(x, y)$ is not a cdf. Alternatively, assume that $G(x, y)$ is a joint cdf for $X$ and $Y$, then

$$P\{ -1 < X \leq 2, -1 < Y \leq 2 \} = G(2, 2) - G(-1, 2) - G(2, -1) + G(-1, -1) = 1 - 1 - 1 + 0 = -1.$$  

But this violates the property that the probability of any event must be nonnegative.

5. **Time until the $n$-th arrival.** Let the random variable $N(t)$ be the number of packets arriving during time $(0, t]$. Suppose $N(t)$ is Poisson with pmf

$$p_N(n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \quad \text{for } n = 0, 1, 2, \ldots.$$  

Let the random variable $Y$ be the time to get the $n$-th packet. Find the pdf of $Y$.

**Solution:** To find the pdf $f_Y(t)$ of the random variable $Y$, note that the event $\{ Y \leq t \}$ occurs iff the time of the $n$th packet is in $(0, t]$, that is, iff the number $N(t)$ of packets arriving in $[0, t]$ is at least $n$. Alternatively, $\{ Y > t \}$ occurs iff $\{ N(t) < n \}$. Hence, the cdf $F_Y(t)$ of $Y$ is given by

$$F_Y(t) = P\{ Y \leq t \} = P\{ N(t) \geq n \} = \sum_{k=n}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$  

Differentiating $F_Y(t)$ with respect to $t$, we get the pdf $f_Y(t)$ as

$$f_Y(t) = \sum_{k=n}^{\infty} \left[ -\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} \right]$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} - \sum_{k=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!} + \sum_{k=n+1}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$  

for $t > 0$.

Or we can use another way. Since we know that the time interval $T$ between packet arrivals is an exponential random variable with pdf

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Let $T_i$ denote the i.i.d. exponential interarrival times, then $Y = T_1 + T_2 + \cdots + T_n$. By convolving $f_T(t)$ with itself $n-1$ times, which can be also computed by its Fourier transform (characteristic function), we can show that the pdf of $Y$ is given by

$$f_Y(t) = \begin{cases} \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, & \text{if } t \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

6. **Diamond distribution.** Consider the random variables $X$ and $Y$ with the joint pdf

$$f_{X,Y}(x, y) = \begin{cases} c & \text{if } |x| + |y| \leq 1/\sqrt{2} \\ 0 & \text{otherwise}, \end{cases}$$

where $c$ is a constant.

(a) Find $c$.

(b) Find $f_X(x)$ and $f_{X|Y}(x|y)$.

(c) Are $X$ and $Y$ independent random variables? Justify your answer.

(d) Define the random variable $Z = (|X| + |Y|)$. Find the pdf $f_Z(z)$.

**Solution:**

(a) The integral of the pdf $f_{X,Y}(x, y)$ over $-\infty < x < \infty$, $-\infty < y < \infty$ is $c$, and therefore by the definition of joint density

$$c = 1.$$

(b) The marginal pdf is obtained by integrating the joint pdf with respect to $y$. For $0 \leq x \leq \frac{1}{\sqrt{2}}$,

$$f_X(x) = \int_{-\frac{1}{\sqrt{2}}-x}^{\frac{1}{\sqrt{2}}-x} c \, dy = 2 \left( \frac{1}{\sqrt{2}} - x \right),$$

and for $-\frac{1}{\sqrt{2}} \leq x \leq 0$,

$$f_X(x) = \int_{-\frac{1}{\sqrt{2}}+x}^{\frac{1}{\sqrt{2}}+x} c \, dy = 2 \left( \frac{1}{\sqrt{2}} + x \right).$$

So the marginal pdf may be written as

$$f_X(x) = \begin{cases} \sqrt{2} - 2|x| & |x| \leq \frac{1}{\sqrt{2}} \\ 0 & \text{otherwise}. \end{cases}$$
Now since \( f_{XY}(x, y) \) is symmetrical, \( f_Y(y) = f_X(y) \). Thus,

\[
f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} \frac{1}{\sqrt{2-2|y|}} & |x| + |y| \leq \frac{1}{\sqrt{2}}, \ |y| \leq \frac{1}{\sqrt{2}} \\ 0 & \text{otherwise.} \end{cases}
\]

(c) \( X \) and \( Y \) are not independent since

\[
f_{X,Y}(x, y) \neq f_X(x)f_Y(y).
\]

Alternatively, \( X \) and \( Y \) are not independent since \( f_{X|Y}(x|y) \) depends on the value of \( y \).

(d) We have, for \( 0 \leq z < 1/\sqrt{2} \),

\[
F_Z(z) = \mathbb{P}(Z \leq z) = \int_{-\infty}^{z} F_X(|X| + |Y| \leq z) \, dy = \int_{-\infty}^{z} P(|X| \leq z - |y| | Y = y) \, f_Y(y) \, dy
\]

\[
= \int_{-z}^{z} P(|Y| \leq |y|) \cdot f_Y(y) \, dy = \int_{-z}^{z} f_{X|Y}(x|y) f_Y(y) \, dx \, dy
\]

\[
= \int_{-z}^{z} \frac{2(z - |y|)}{\sqrt{2} - 2|y|} \left( \sqrt{2} - 2|y| \right) \, dy = 4 \int_{0}^{z} (z - y) \, dy = 2z^2.
\]

Thus, \( f_Z(z) = \begin{cases} 4z, & z \in (0, 1/\sqrt{2}) \\ 0, & \text{otherwise.} \end{cases} \)

7. **Coin with random bias.** You are given a coin but are not told what its bias (probability of heads) is. You are told instead that the bias is the outcome of a random variable \( P \sim \text{Unif}[0, 1] \). To get more information about the coin bias, you flip it independently 10 times. Let \( X \) be the number of heads you get. Thus \( X \sim \text{B}(10, P) \). Assuming that \( X = 9 \), find and sketch the a posteriori probability of \( P \), i.e., \( f_{P|X}(p|x) \).

**Solution:** In order to find the conditional pdf of \( P \), apply Bayes’ rule for mixed random variables to get

\[
f_{P|X}(p|x) = \frac{p_{X|P}(x|p)}{p_X(x)} f_P(p) = \frac{p_{X|P}(x|p)}{\int_{0}^{1} p_{X|P}(x|p) f_P(p) \, dp} f_P(p).
\]
Now it is given that $X = 9$, thus for $0 \leq p \leq 1$

\[
f_{P|X}(p|9) = \frac{p^9(1-p)}{\int_0^1 p^9(1-p) \, dp}
= \frac{p^9(1-p)}{110}
= 110p^9(1-p).
\]

Figure 3 compares the unconditional and the conditional pdfs for $P$. It may be seen that given the information that 10 independent tosses resulted in 9 heads, the pdf is shifted towards the value $\frac{9}{10}$.

8. **Functions of exponential random variables.** Let $X$ and $Y$ be independent exponentially distributed random variables with the same parameter $\lambda$. Define the following three functions of $X$ and $Y$:

\[U = \max(X, Y), \quad V = \min(X, Y), \quad W = U - V.\]

(a) Find the joint pdf of $U$ and $V$.

(b) Find the joint pdf of $V$ and $W$. Are they independent?

**Hint:** You can solve part (b) either directly by finding the joint cdf or by expressing the joint pdf in terms of $f_{U,V}(u,v)$ and using the result of part (a).

**Solution:**
(a) We have, for $0 \leq v \leq u < \infty$,

$$F_{U,V}(u,v) = P(U \leq u, V \leq v)$$

$$= P(U \leq u) - P(U \leq u, V > v)$$

$$= P(\max(X,Y) \leq u) - P(\max(X,Y) \leq u, \min(X,Y) > v)$$

$$= P(X \leq u, Y \leq u) - P(X \in (v,u], Y \in (v,u])$$

$$= [1 - \exp(-\lambda u)]^2 - [\exp(-\lambda v) - \exp(-\lambda u)]^2$$

For $0 \leq u < v < \infty$, we have

$$F_{U,V}(u,v) = P(U \leq u, V \leq v)$$

$$= P(U \leq u)$$

$$= [1 - \exp(-\lambda u)]^2.$$

If any one (or both) of $u$ and $v$ is less than zero, we have $F_{U,V}(u,v) = 0$.

Differentiating $F_{U,V}$, we thus have

$$f_{U,V}(u,v) = \frac{\partial^2 F}{\partial u \partial v}$$

$$= \begin{cases} 2\lambda^2 \exp[-\lambda(u + v)], & 0 \leq v \leq u < \infty \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that $f_{U,V}$ integrates to 1.

Integrating the joint density, we also see that

$$f_U(u) = \begin{cases} 2\lambda e^{-\lambda u}(1 - e^{-\lambda u}), & u \geq 0 \\ 0, & u < 0 \end{cases}, \text{ and}$$

$$f_V(v) = \begin{cases} 2\lambda e^{-2\lambda v}, & v \geq 0 \\ 0, & v < 0 \end{cases}.$$

(b) Using the results of the last part, we note that

$$f_{U|V}(u|v) = \frac{f_{U,V}(u,v)}{f_V(v)} = \begin{cases} \lambda e^{-\lambda(u-v)}, & u \geq v \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

We have, for $v \geq 0$ and $w \geq 0$,

$$F_{V,W}(v,w) = P(V \leq v, W \leq w)$$

$$= P(V \leq v, U \leq V + w)$$

$$= \int P(V \leq v, U \leq V + w | V = v') f_V(v') dv'$$

$$= \int_{v'=v}^{v'=v+w} P(U \leq v' + w | V = v') f_V(v') dv'.$$
Using the usual rule for differentiating an integral, we have
\[
\frac{\partial F_{V,W}}{\partial v} = P\left(U \leq v + w \mid V = v\right)f_V(v).
\]
Differentiating w.r.t. \(w\), we now have
\[
f_{V,W}(v, w) = \frac{\partial^2 F_{V,W}}{\partial w \partial v} = f_U(v + w \mid v) f_V(v) = \lambda \exp[-\lambda(v + w - v)] \cdot 2\lambda \exp(-2\lambda v) = 2\lambda^2 e^{-2\lambda v - \lambda w},
\]
for \(v \geq 0\) and \(w \geq 0\).

Since the density can be written as \(f_{V,W}(v, w) = f_V(v)f_W(w)\) (Check!), \(V\) and \(W\) are independent.

9. First available teller. Consider a bank with two tellers. The service times for the tellers are independent exponentially distributed random variables \(X_1 \sim \text{Exp}(\lambda_1)\) and \(X_2 \sim \text{Exp}(\lambda_2)\) respectively. You arrive at the bank and find that both tellers are busy but that nobody else is waiting to be served. You are served by the first available teller once he/she is free.

(a) What is the probability that you are served by the first teller?
(b) Let the random variable \(Y\) denote your waiting time. Find the pdf of \(Y\).

Solution:

(a) From the memoryless property of the exponential distribution, the remaining services for the tellers are also independent exponentially distributed random variables with parameters \(\lambda_1\) and \(\lambda_2\), respectively. The probability that you will be served by the first teller is the probability that the first teller finishes the service before the second teller does. Thus,
\[
P\{X_1 < X_2\} = \int_{x_2 > x_1} f_{X_1,X_2}(x_1, x_2) \, dx_2 \, dx_1 = \int_{x_1=0}^{\infty} \int_{x_2=x_1}^{\infty} \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2} \, dx_2 \, dx_1 = \int_{x_1=0}^{\infty} \lambda_1 e^{-(\lambda_1 + \lambda_2) x_1} \, dx_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

(b) First observe that \(Y = \min(X_1, X_2)\). Since
\[
P\{Y > y\} = P\{X_1 > y, X_2 > y\} = P\{X_1 > y\}P\{X_2 > y\} = e^{-\lambda_1 y} \times e^{-\lambda_2 y} = e^{-(\lambda_1 + \lambda_2) y}
\]
for \( y \geq 0 \), \( Y \) is an exponential random variable with pdf
\[
f_Y(y) = \begin{cases} 
(\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)y}, & y \geq 0, \\
0, & \text{otherwise.}
\end{cases}
\]

10. Two independent uniform random variables.
Let \( X \) and \( Y \) be independently and uniformly drawn from the interval \([0, 1]\).

(a) Find the pdf of \( U = \max(X, Y) \).
(b) Find the pdf of \( V = \min(X, Y) \).
(c) Find the pdf of \( W = U - V \).
(d) Find the probability \( P\{|X - Y| \geq 1/2\} \).

Solution:

(a) We have
\[
F_U(u) = P\{U \leq u\} = P\{\max(X, Y) \leq u\} = P\{X \leq u, Y \leq u\} = P\{X \leq u\}P\{Y \leq u\} = u^2
\]
for \( 0 \leq u \leq 1 \). Hence,
\[
f_U(u) = \begin{cases} 
2u, & 0 \leq u \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]
(b) Similarly,
\[
1 - F_V(v) = P\{V > v\} = P\{\min(X, Y) > v\} = P\{X > v, Y > v\} = P\{X > v\}P\{Y > v\} = (1 - v)^2,
\]
or equivalently, \( F_V(v) = 1 - (1 - v)^2 \), for \( 0 \leq v \leq 1 \). Hence,
\[
f_V(v) = \begin{cases} 
2(1 - v), & 0 \leq v \leq 1, \\
0, & \text{otherwise.}
\end{cases}
\]
(c) First note that \( W = U - V = |X - Y| \). (Why?) Hence,
\[
P\{W \leq w\} = P\{|X - Y| \leq w\} = P(-w \leq X - Y \leq w).
\]
Since \( X \) and \( Y \) are uniformly distributed over \([0, 1]\), the above integral is equal to the area of the shaded region in the following figure:
The area can be easily calculated as \(1 - (1 - w)^2\) for \(0 \leq w \leq 1\). Hence \(F_W(w) = 1 - (1 - w)^2\) and
\[
f_W(w) = \begin{cases} 
2(1 - w), & 0 \leq w \leq 1, \\
0, & \text{otherwise}.
\end{cases}
\]

(d) From the figure above,
\[
P\{|X - Y| \geq 1/2\} = P\{W \geq 1/2\} = 1/4.
\]
11. Let $X$ and $Y$ be independent Gaussians, both with zero mean and unit variance. Find the pdf of $|X - Y|$.

**Solution:** Since $X$ and $Y$ are independent Gaussians, they are jointly Gaussian and hence, $X - Y$ is also Gaussian.

We can calculate the pdf of $U = X - Y$ by the convolution of the pdfs of $X$ and $-Y$ (the latter being the same as the pdf of $Y$).

We thus have

$$f_U(u) = \int_{-\infty}^{\infty} f_X(x)f_Y(u-x)dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[ -\frac{x^2 + (u-x)^2}{2} \right] dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[ -\frac{x^2 + ux - u^2}{2} \right] dx$$

$$= \frac{1}{2\sqrt{\pi}} \exp \left[ -\frac{u^2}{4} \right].$$

Thus, $U \sim \mathcal{N}(0, 2)$. This can also be realized more easily by noting that


Let $V = |U|$.

Then for $v < 0$, $F_V(v) = P(V \leq v) = 0$.

For $v \geq 0$, we have

$$F_V(v) = P(V \leq v)$$

$$= P(\{|U| \leq v\})$$

$$= P(-v \leq U \leq v)$$

$$= F_U(v) - F_U(-v)$$

(since $U$ is a continuous random variable)

$$= \int_{-v}^{v} f_U(u)du$$

$$= 2 \int_{0}^{v} f_U(u)du$$

(since $f_U(u)$ is symmetric).

Differentiating this expression, we see that $f_V(v) = 2f_U(v)$ for $v \geq 0$. 

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Thus the pdf of \( V = |X - Y| \) is given by
\[
f_V(v) = \begin{cases} 
\frac{1}{\sqrt{\pi}} \exp \left[ -\frac{v^2}{4} \right], & v \geq 0 \\
0, & \text{otherwise}.
\end{cases}
\]

It is easy to verify that the density integrates to 1, as required.

12. Maximal correlation.

(a) For any pair of random variables \((X, Y)\), show that
\[
F_{X,Y}(x, y) \leq \min \{ F_X(x), F_Y(y) \}.
\]

Now let \( F \) and \( G \) be continuous and invertible cdf’s and let \( X \sim F \).

(b) Find the distribution of \( Y = G^{-1}(F(X)) \).

(c) Show that
\[
F_{X,Y}(x, y) = \min \{ F(x), G(y) \}.
\]

Solution:

(a) We have
\[
F_{X,Y}(x, y) = P\{X \leq x, Y \leq y\} \leq P\{X \leq x\} = F_X(x),
\]
and similarly, \( F_{X,Y} \leq F_Y(y) \). Thus,
\[
F_{X,Y} \leq \min \{ F_X(x), F_Y(y) \}.
\]

(b) We have
\[
F_Y(y) = P\{Y \leq y\} = P\{G^{-1}(F(X)) \leq y\} = P\{F(X) \leq G(y)\} = P\{X \leq F^{-1}(G(y))\} = F(F^{-1}(G(y))) = G(y).
\]

(c) We have
\[
F_{X,Y}(x, y) = P\{X \leq x, Y \leq y\} = P\{X \leq x, X \leq F^{-1}(G(y))\} = P\{X \leq \min\{x, F^{-1}(G(y))\}\} = \min\{F(x), F(F^{-1}(G(y)))\} = \min\{F(x), G(y)\}.
\]

From part (a), this is the maximal joint cdf for any \((X, Y)\) with the given marginal cdf’s \( F(x) \) and \( G(y) \).