Practice Final Examination (Winter 2015)

There are 6 problems, each problem with multiple parts, each part worth 10 points. Your answer should be as clear and readable as possible. Justify any claim that you make.

1. Drawing balls without replacement (20 pts). Suppose that we have an urn containing one red ball and \( n - 1 \) white balls. Each time we draw a ball at random from the urn without replacement (so after the \( n \)-th drawing, there is no ball left in the urn). For \( i = 1, 2, \ldots, n \), let
   \[
   X_i = \begin{cases} 
   1 & \text{if the } i\text{-th ball is red}, \\
   0 & \text{otherwise}.
   \end{cases}
   \]
   (a) Find \( E[X_i] \), \( i = 1, 2, \ldots, n \).
   (b) Find \( \text{Var}(X_i) \) and \( \text{Cov}(X_i, X_j) \), \( i, j = 1, 2, \ldots, n \).

2. Correlation coefficients (30 pts). Let \( X_1, X_2, X_3 \) be three identically distributed, but not necessarily independent random variables with zero mean and unit variance. Let \( \rho_{ij} \) be the correlation coefficient between \( X_i \) and \( X_j \) for \( i \neq j \in \{1, 2, 3\} \).
   (a) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = 1 \)? If so, construct a random triple with such correlation coefficients. If not, justify why not.
   (b) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = -1 \)? If so, construct a random triple with such correlation coefficients. If not, justify why not.
   (c) Is it possible to have \( \rho_{12} = \rho_{13} = \rho_{23} = -1/2 \)? If so, construct a random triple with such correlation coefficients. If not, justify why not.

3. Sampled random walk (30 pts). Let \( \{X_n\} \) be the (standard) symmetric random walk, i.e.,
   \[
   X_0 = 0,
   X_n = \sum_{i=1}^{n} Z_i, \quad n = 1, 2, \ldots,
   \]
   where \( Z_1, Z_2, \ldots \) are i.i.d. with \( P\{Z_1 = -1\} = P\{Z_1 = 1\} = 1/2 \). Let \( \{Y_n\} \) be a sampled version of \( \{X_n\} \) defined by
   \[
   Y_n = X_{2n}, \quad n = 0, 1, 2, \ldots.
   \]
   (a) Is \( \{Y_n\} \) independent increment? Justify your answer.
   (b) Is \( \{Y_n\} \) Markov? Justify your answer.
   (c) Find \( E[Y_3 | Y_2] \).
4. Random binary modulation (20 pts). Let \( \{X_n\} \) be a zero-mean wide-sense stationary random process with autocorrelation function \( R_X(n) \), and \( Z_1, Z_2, \ldots \) be i.i.d. Bern\( (p) \) random variables, i.e.,

\[
Z_i = \begin{cases} 
1, & \text{with probability } p, \\
0, & \text{with probability } 1 - p.
\end{cases}
\]

Assume that \( \{X_n\} \) and \( \{Z_n\} \) are independent. Let \( Y_n = X_n \cdot Z_n, \quad n = 1, 2, \ldots \).

(a) Find the mean and the autocorrelation function of \( \{Y_n\} \) in terms of \( R_X(n) \) and \( p \).

(b) Is \( \{Y_n\} \) jointly wide-sense stationary with \( \{X_n\} \)?

5. Wiener process (30 pts). Recall the following definition of the (standard) Wiener process:

- \( W(0) = 0 \),
- \( \{W(t)\} \) is independent increment with \( W(t) - W(s) \sim N(0, t - s) \) for all \( t > s \),
- \( P\{\omega : W(\omega, t) \text{ is continuous in } t\} = 1 \).

Let \( W_1(t) \) and \( W_2(t) \) be independent Wiener processes.

(a) Find the mean and the variance of

\[
X(t) = \frac{1}{\sqrt{2}} (W_1(t) + W_2(t)).
\]

Is \( \{X(t)\} \) a Wiener process? Justify your answer.

(b) Find the mean and the variance of

\[
Y(t) = \frac{1}{\sqrt{2}} (W_1(t) - W_2(t)).
\]

Is \( \{Y(t)\} \) a Wiener process? Justify your answer.

(c) Find \( E[X(t)Y(s)] \).

6. Derivatives of stochastic processes (60 points). Let \( \{X(t)\} \) be a wide-sense stationary random process with mean zero and autocorrelation function \( R(\tau) = e^{-|\tau|} \). Recall that a random process \( \{Y(t)\} \) is continuous in mean square if \( E[(Y(t + \epsilon) - Y(t))^2] \to 0 \) as \( \epsilon \to 0 \).

(a) Find the mean and the variance of \( X(t) \).

(b) Is \( X(t) \) continuous in mean square? Justify your answer.

(c) Now let

\[
Z_\epsilon(t) = \frac{X(t + \epsilon) - X(t)}{\epsilon}
\]

be an \( \epsilon \)-approximation of the derivative \( \dot{X}(t) \). Find the mean and the variance of \( Z_\epsilon(t) \).

(d) Find the linear MMSE estimate of \( Z_\epsilon(t) \) given \( (X(t), X(t + \epsilon)) \) and the associated MSE.

(e) Find the linear MMSE estimate of \( Z_\epsilon(t) \) given \( X(t) \) and the associated MSE.

(f) Find the limiting mean and variance of \( Z_\epsilon(t) \) as \( \epsilon \to 0 \).