Solutions to Practice Midterm

There are 4 problems, each problem with multiple parts, each part worth 10 points. Your answer should be as clear and readable as possible. Please justify any claim that you make. The following facts might be useful:

\[ \int e^{-\alpha t} \, dt = -\frac{1}{\alpha} e^{-\alpha t} + c, \]
\[ \int t^k \, dt = \frac{1}{k+1} t^{k+1} + c. \]

1. Unfair game (10 pts). Two players A and B play the following game. Player A is given a random number distributed uniformly from \([0, 9]\) and player B is given a random number distributed uniformly from \([0, 10]\). We assume that the numbers are independent. The person who gets the larger number will win the game. Find the probability that player B wins the game.

**Solution:** Let \(X\) and \(Y\) be the numbers given to players A and B respectively. Hence \(X \sim \text{Unif}[0, 9]\) and \(Y \sim \text{Unif}[0, 10]\) and player B wins if and only if \(X \leq Y\). By symmetry we have

\[ P\{X \leq Y \mid Y \leq 9\} = \frac{1}{2}. \]

Therefore, using the law of total probability we have

\[ P\{B \text{ wins}\} = P\{X \leq Y\} = P\{X \leq Y \mid Y > 9\} P\{Y > 9\} + P\{X \leq Y \mid Y \leq 9\} P\{Y \leq 9\} \]

\[ = 1 \times \frac{1}{10} + \frac{1}{2} \times \frac{9}{10} = \frac{55}{100}. \]

Alternatively,

\[ P\{X \leq Y\} = \int_0^9 \int_x^{10} f_{X,Y}(x, y) \, dy \, dx = \int_0^9 \int_x^{10} \frac{1}{9} \frac{1}{10} \, dy \, dx = \frac{1}{90} \int_0^9 (10 - x) \, dx = \frac{55}{100}. \]

2. Coin with random bias (40 pts). A random variable \(P\) is drawn according to the following distribution

\[ f_P(p) = \begin{cases} 2p & \text{if } 0 \leq p \leq 1, \\ 0 & \text{otherwise}. \end{cases} \]

Then a coin with bias \(P\) is flipped \(n\) times. Assume that the value of the bias does not change during the sequence of tosses.
(a) Find the probability that the first flip is heads.

(b) Find the probability that the first flip is heads and the second flip is tails.

(c) Find the probability that the third flip is heads given that the first flip is heads and the second flip is tails.

(d) Find the expected number of heads in \(n\) flips.

**Solution:** Let \(X_i, i = 1, 2, 3\), denote the outcome of the \(i\)-th coin flip.

(a) By the law of total probability

\[
P\{X_1 = H\} = \int_0^1 P\{X_1 = H \mid P = p\} f_P(p) \, dp = \int_0^1 p \times 2p \, dp = \frac{2}{3}.
\]

(b) Given \(P = p\) the coin flips are independent of each other. Therefore, by the law of total probability we have

\[
P\{X_1 = H, X_2 = T\} = \int_0^1 P\{X_1 = H, X_2 = T \mid P = p\} f_P(p) \, dp
\]

\[
= \int_0^1 P\{X_1 = H \mid P = p\} P\{X_2 = T \mid P = p\} f_P(p) \, dp
\]

\[
= \int_0^1 p(1-p) \times 2p \, dp = \frac{1}{6}.
\]

(c) We have

\[
P\{X_3 = H \mid X_1 = H, X_2 = T\} = \frac{P\{X_3 = H, X_2 = T, X_1 = H\}}{P\{X_1 = H, X_2 = T\}}.
\]

By the law of total probability we have

\[
P\{X_3 = H, X_2 = T, X_1 = H\} = \int_0^1 P\{X_3 = H, X_2 = T, X_1 = H \mid P = p\} f_P(p) \, dp
\]

\[
= \int_0^1 p^2(1-p) \times 2p \, dp = \frac{1}{10}.
\]

Therefore,

\[
P\{X_3 = H \mid X_1 = H, X_2 = T\} = \frac{\frac{1}{10}}{\frac{1}{6}} = \frac{6}{10}.
\]

(d) Let \(N\) be the number of heads in \(n\) flips. Given \(P = p\), \(N\) is a binomial random variable, i.e., \(N\mid_{P=p} \sim \text{Binom}(n, p)\). Therefore

\[
E[N \mid P = p] = np,
\]

and hence by the law of iterated expectation

\[
E[N] = E[E[N \mid P]] = E[nP] = nE[P] = n \int_{-\infty}^{\infty} p f_P(p) \, dp = n \int_0^1 p \times 2p \, dp = \frac{2}{3}n.
\]
3. Channel with Laplacian noise (20 pts). Let the signal
\[ X = \begin{cases} +1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}, \end{cases} \]
and the noise \( Z \sim f_Z(z) = \frac{1}{2}e^{-|z|} \) be independent random variables. Their sum \( Y = X + Z \) is observed. We wish to detect \( X \) based on the observation \( Y \).

(a) Find the function \( g^*(y) \) that minimizes the probability of error \( P\{X \neq g(Y)\} \).
(b) Find the minimum probability of error achieved by \( g^*(y) \) in part (a).

Solution:
(a) By the MAP rule we have
\[
g^*(y) = \begin{cases} +1 & \text{if } P\{X = 1 | Y = y\} \geq P\{X = -1 | Y = y\}, \\ -1 & \text{otherwise.} \end{cases}
\]
Since \( P\{X = 1\} = P\{X = -1\} \), the MAP rule is equivalent to the ML rule:
\[
g^*(y) = \begin{cases} +1 & \text{if } f_{Y|X}(y|1) \geq f_{Y|X}(y|-1), \\ -1 & \text{otherwise.} \end{cases}
\]
We have
\[
f_{Y|X}(y|1) = f_{Z|X}(y-1|1) = f_Z(y-1),
\]
and
\[
f_{Y|X}(y|-1) = f_{Z|X}(y+1|-1) = f_Z(y+1).
\]
These two conditional densities are shown in Figure 1.

![Conditional probability density functions](image)

Figure 1: Conditional probability density functions \( f_{Y|X}(y|1) \) and \( f_{Y|X}(y|-1) \).

By comparing the two conditional densities we have
\[
g^*(y) = \begin{cases} +1 & \text{if } y \geq 0, \\ -1 & \text{if } y < 0. \end{cases}
\]
(b) We have

\[
P\{X \neq D(Y)\} = P\{X = 1, D(Y) = -1\} + P\{X = -1, D(Y) = +1\}
\]

\[
= P\{X = 1\}P\{Y < 0 | X = 1\} + P\{X = -1\}P\{Y \geq 0 | X = -1\}
\]

\[
= \frac{1}{2}P\{Z < -1\} + \frac{1}{2}P\{Z \geq 1\}
\]

\[
= P\{Z \geq 1\}
\]

\[
= \int_{1}^{\infty} \frac{1}{2}e^{-z} \, dz
\]

\[
= \frac{1}{2}e.
\]

4. Exponential random variables (30 pts). Let \(X\) and \(Y\) be two iid exponential random variables according to the following densities.

\[
f_X(x) = \begin{cases} 
  e^{-x} & \text{if } x \geq 0, \\
  0 & \text{otherwise,}
\end{cases}
\]

and

\[
f_Y(y) = \begin{cases} 
  e^{-y} & \text{if } y \geq 0, \\
  0 & \text{otherwise.}
\end{cases}
\]

(a) Find the pdf of \(U = \max\{X, Y\}\).

(b) Find the conditional pdf of \(X\) given the event \(\{X \geq Y\}\).

(c) Find the joint cdf of \(X\) and \(U\), \(F_{U,X}(u, x)\).

**Solution:** we have

\[
F_X(x) = \int_{0}^{x} f_X(u) \, du = \int_{0}^{x} e^{-u} \, du = 1 - e^{-x}.
\]

Similarly,

\[
F_Y(y) = 1 - e^{-y}.
\]

(a) Since \(X\) and \(Y\) are independent, for \(u \geq 0\) we have

\[
F_U(u) = P\{U \leq u\} = P\{\max\{X, Y\} \leq u\} = P\{X \leq u, Y \leq u\}
\]

\[
= P\{X \leq u\}P\{Y \leq u\} = (1 - e^{-u})^2.
\]

Therefore,

\[
f_U(u) = \frac{d}{du}F_U(u) = \begin{cases} 
  2(1 - e^{-u})e^{-u} & \text{if } u \geq 0, \\
  0 & \text{otherwise.}
\end{cases}
\]

(b) For \(x \geq 0\), we have

\[
P\{X \leq x | X \geq Y\} = \frac{P\{X \leq x, X \geq Y\}}{P\{X \geq Y\}}.
\]
By symmetry, $P\{X \geq Y\} = \frac{1}{2}$.

\[
P\{X \leq x, X \geq Y\} = \int_0^x \int_0^t f_{X,Y}(t, y) \, dy \, dt
\]
\[
= \int_0^x \int_0^t e^{-t}e^{-y} \, dy \, dt
\]
\[
= \int_0^x e^{-t}(1 - e^{-t}) \, dt
\]
\[
= -e^{-x} + \frac{1}{2}e^{-2x} + \frac{1}{2}
\]
\[
= \frac{1}{2}(1 - e^{-x})^2.
\]

Alternatively, by symmetry

\[
P\{X \leq x, X \geq Y\} = \frac{1}{2}P\{X \leq x, Y \leq x\} = \frac{1}{2}(1 - e^{-x})^2.
\]

Therefore,

\[
P\{X \leq x \mid X \geq Y\} = \frac{1}{2}(1 - e^{-x})^2 = (1 - e^{-x})^2.
\]

Let $A$ be the event $\{X \geq Y\}$. Then for $x \geq 0$

\[
f_{X\mid A}(x) = 2(1 - e^{-x})e^{-x}.
\]

Note that the answer is similar to the answer of part (a).

(c)

\[
F_{U,X}(u, x) = P\{U \leq u, X \leq x\}
\]
\[
= P\{Y \leq u, X \leq u, X \leq x\}
\]
\[
= P\{Y \leq u, X \leq \min\{u, x\}\}
\]
\[
= P\{Y \leq u\}P\{X \leq \min\{u, x\}\}
\]
\[
= (1 - e^{-u})(1 - e^{-\min\{u, x\}})
\]
\[
= \begin{cases} 
(1 - e^{-u})^2 & \text{if } x \geq u, \\
(1 - e^{-u})(1 - e^{-x}) & \text{if } x < u.
\end{cases}
\]