FREQUENCY RESPONSE BODE PLOTS

Last lecture we have focused on the Frequency Response \( H(\omega) \) which is a complex quantity that describes the behavior of the system under sinusoidal excitation.

Bode Plots are a convenient graphical representation of the frequency response which makes it possible to appreciate exceedingly small values of the response (see attached figures).

The trick is to represent the response on a logarithmic scale rather than on a linear scale.

We need to plot \( |H(\omega)| \) and \( \angle H(\omega) \) separately.

We let \( \omega \) vary on a logarithmic scale, which means that on the \( \omega \)-axis we report decades.

\[
\begin{align*}
10 & \quad 100 & \quad 10^3 & \quad 10^4 & \cdots & \omega
\end{align*}
\]

Then for \( \angle H(\omega) \) we report the angle radians (or degrees), while for \( |H(\omega)| \) we report dB's which is \( 20 \log_{10} |H(\omega)| \).
Let us start with our favorite low-pass filter:

\[ \frac{1}{1 + j \omega RC} = H(\omega) \]

\[ |H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \]

\[ 20 \log |H(\omega)| = -20 \log \sqrt{1 + (\omega RC)^2} \]

Let \( \frac{1}{RC} = \omega_0 \) Then we have

\[ 20 \log |H(\omega)| = -20 \log \sqrt{1 + (\omega/\omega_0)^2} \]

We study two asymptotic regimes

1) \( \omega << \omega_0 \) \( \Rightarrow \) 
\[ -20 \log |H(\omega)| \propto -20 \log 1 = 0 \text{ dB} \]

2) \( \omega >> \omega_0 \) \( \Rightarrow \) 
\[ -20 \log |H(\omega)| \propto -20 \log \sqrt{(\omega_0^2)} = -20 \log \omega_0 \]

\[ \text{ which is a line of constant slope } -20 \text{ dB/} \text{decade} \]

\[ \text{ in } \text{dB scale} \]

\[ \text{slope is } -20 \text{ dB/} \text{decade} \]

**Question:** what happens when \( \omega = \omega_0 \)?
The situation is analogous in the study of the phase.
The trick is always to identify the behavior in the asymptotic regime.
From last lecture we know that the phase shift will be close to zero for \( \omega < \omega_0 \) and will tend to \(-\pi/2\) and for \( \omega = \omega_0 \) the phase is \(-\pi/4\).

So approximately we say that the phase starts shifting one decade before \( \omega_0 \) and has shifted by \(-\pi/2\) one decade after \( \omega_0 \).

**Summary**

For magnitude \( |H(\omega)| \)
- identify the "breakpoint" \( \omega_0 \)
- until \( \omega_0 \) the magnitude is 0 dB
- after \( \omega_0 \) the magnitude is a line of constant slope 20 dB/decade.

For phase \( \angle H(\omega) \)
- identify the "breakpoint" \( \omega_0 \)
- the phase shift starts one decade before \( \omega_0 \)
- the phase shift is practically complete one decade after \( \omega_0 \)
Now, we can generalize this asymptotic method to more complicated expressions of the frequency response. Let's look at an example.

\[ H(w) = \frac{0.1 \text{j}w + 20}{2 \cdot 10^{-5} (\text{j}w)^3 + 0.1002 (\text{j}w)^2 + \text{j}w} \]

First, we write the equation in standard form.

\[ H(w) = \frac{0.1 \text{j}w + 20}{2 \cdot 10^{-5} (\text{j}w)^3 + 0.1002 (\text{j}w)^2 + \text{j}w} = \frac{20 (\text{j}w/200 + 1)}{\text{j}w (\text{j}w/10 + 1)(\text{j}w/5000 + 1)} \]

Now we consider all the factors separately. This is possible because:

\[ \lim_{\omega \to 0} \log H(\omega) = 20 \log 20 + 20 \log (1 + \text{j}w/100) - 20 \log \text{j}w - 20 \log (1 + \text{j}w/5000) \]

For each of these terms we write an asymptotic Bode plot like the one we have seen for the low-pass filter, and then at the end we sum all of them. Let's look at the magnitude first.

\[ \begin{align*}
20 \log 20 \\
20 \\
10 \\
10^2 \\
10^3 \\
10^4 \\
10^5 \\
\end{align*} \]
We combine all terms in a single plot.

For phase is similar.

First find points:
- $40 \rightarrow -45^\circ$ (A)
- $200 \rightarrow +45^\circ$ (B)
- $5000 \rightarrow -45^\circ$ (C)

The phase shift starts one decade before and ends one decade after points A, B, and C.

- A: $10 \rightarrow -45^\circ$ [$1, 100$]
- B: $200 \rightarrow +45^\circ$ [$20, 2000$]
- C: $5000 \rightarrow -45^\circ$ [$500, 5 \times 10^4$]

Intervals of interest are:
- $[1, 20]$ [$20, 100$] [$100, 500$]
- $[500, 2000]$ [$2000, 5 \times 10^4$]

[/1,20] $\rightarrow$ A decrease

[/20,100] $\rightarrow$ A+B constant

[/100,500] $\rightarrow$ B increase

[/500,2000] $\rightarrow$ Btc constant

[/2000,5\times10^4] $\rightarrow$ C decrease

10° 100° 20° 500° 2000°, 5 \times 10^4°
\[ H(\omega) = \frac{0.1 j\omega + 20}{2 \times 10^{-5} (j\omega)^3 + 0.1002 (j\omega)^2 + j\omega} \]

**Figure 6.41** Approximate (asymptotic) frequency response of individual first-order terms. (a) Magnitude; (b) Phase
$$H(\omega) = \frac{0.1 \omega + 20}{2 \cdot 10^{-5} (\omega^3) + 0.1002 (\omega^2) + \omega}$$

**Figure 6.42** Comparison of approximate and exact frequency response. (a) Magnitude; (b) phase
5 Frequency response $H(\omega)$ of LTI systems

Figure 5.7: Linear (a, c) and decibel or dB (b, d) plots of the amplitude response of systems $H(\omega) = \frac{1}{1+j\omega}$ (a, b) and $H(\omega) = \frac{1}{(1+j\omega)(100+j\omega)}$ (c, d). Note that in dB plots a logarithmic scale is used for the horizontal axes following a common engineering practice.

Using Bode Plots
It is possible to appreciate exceedingly small values of the response.