In last lecture we have seen a way to represent periodic signals in terms of Fourier Series, which is a convenient representation to find the response of the system using the principle of superposition and the frequency response.

What we have done:

1. Assume we can write

   \[ x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \]

   \( \omega_0 \) is the fundamental frequency of the periodic signal \( x(t) \)

   \( c_n \) are the harmonic components at frequency \( n\omega_0 \) of the signal \( x(t) \)

   \( c_k \) are the coefficients of the series expansion.

2. We have found a formula to compute the coefficient \( c_k \)

   \[ c_k = \frac{1}{T} \int_{0}^{T} x(t) e^{-jkw_0 t} \, dt \]
PARSEVAL THEOREM

\[ \frac{1}{T} \int |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \]

Proof

\[ |x(t)|^2 = x(t) \, x^*(t) \]
\[ x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{j k \omega_0 t} \]
\[ x^*(t) = \sum_{k=-\infty}^{\infty} c_k^* \, e^{-j k \omega_0 t} \]

\[ \frac{1}{T} \int |x(t)|^2 \, dt = \frac{1}{T} \int \left( \sum_{k=-\infty}^{\infty} c_k \, e^{j k \omega_0 t} \right) \cdot \left( \sum_{m=-\infty}^{\infty} c_m^* \, e^{-j m \omega_0 t} \right) \, dt \]

\[ = \frac{1}{T} \int \sum_{k} \sum_{m} c_k c_m^* e^{j (k-m) \omega_0 t} \, dt \]

\[ = \frac{1}{T} \int \sum_{k} c_k c_k^* + \sum_{k} \sum_{m \neq k} c_k c_m^* e^{j (k-m) \omega_0 t} \, dt \]

\[ = \sum_{k=-\infty}^{\infty} |c_k|^2 + \sum_{k} \sum_{m \neq k} \left( \frac{1}{T} \int e^{j (k-m) \omega_0 t} \, dt \right) \]

The avg power of the signal is conserved in its harmonic components.
Notice that in the application of the proof of Parseval Theorem we have discovered an important property of the Fourier coefficients. That is:

If a signal \( x(t) \) has Fourier coeff. \( c_k \)
then \( x(t)^* \) has Fourier coeff. \( c^*_k \)

From which we also have:
if \( x(t) \) is real then \( x(t) = x^*(t) \)
so that \( c_k = c^*_k \)

A similar property is Time reversed:
If a signal \( x(t) \) has Fourier coeff. \( c_k \)
then \( x(-t) \) has Fourier coeff. \( c_{-k} \)

From which we also have:
if \( x(t) \) is even then \( x(t) = x(-t) \)
then \( c_k = c_{-k} \)

if \( x(t) \) is odd then \( x(t) = -x(-t) \)
then \( c_k = -c_{-k} \)
Another property that we can see from the proof of Parseval's Theorem is the multiplication property

\[ x(t) = \sum_k c_k e^{jk\omega t} \]
\[ y(t) = \sum_m c_m e^{jm\omega t} \]
\[ x(t) y(t) = \sum_k \sum_m c_k c_m^* e^{i(k+m)\omega t} \]

Let \( k + m = l \)
\[ m = l - k \]
\[ x(t) y(t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k c_{l-k}^* e^{il\omega t} \]
\[ = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k c_{l-k}^* e^{il\omega t} \]
\[ \overline{C_l} = \sum_{k=-\infty}^{\infty} c_k c_{l-k}^* \]

The Fourier coefficients of the product of two Fourier Series are given by the above expression, which is called the discrete convolution of the coefficients.
Finally, we have the time-shift property

\[ x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \]

\[ y(t) = x(t - t_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 (t - t_0)} \]

\[ = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} e^{-jk\omega_0 t_0} \]

\[ = \sum_{k=-\infty}^{\infty} c_k \]

if we shift the signal in time.

The coeff. \( c_k \) has a phase shift.

And the time-scaling

\[ x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \]

\[ y(t) = x(at) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 at} \]

\[ \text{Lo-rig. is scaled by the tempo factor} \]
You might have noticed (and I told you to be careful!) that in the computation of the coefficients of the Fourier series and in some of the properties we are freely exchanging the order of the series and integral on the order of two series.

This can be done only if we assume a certain kind of convergence of the Fourier series to the periodic function \( x(t) \) and has been source of debate of mathematicians for many years.

Let's see what we are doing.

- We are assuming we can write a periodic signal
  \[
x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}
  \]
  which means that the series converges to \( x(t) \) in some well-defined sense.

- Let \( c_k e^{jkw_0 t} = f(k, t) \), in the computations we're often doing the following:
  \[
  \frac{1}{T} \int_{x} f(k, t) dt = \sum_{k} \frac{1}{T} \int_{x} f(k, t) dt
  \]
  This is allowed only if the series converges in some well-defined sense.
  More precisely, a theorem by Fabini states that this inversion operation is allowed if:
  \[
  \frac{1}{T} \int_{x} \left| \sum_{k=-\infty}^{\infty} |c_k e^{jkw_0 t}| dt < \infty
  \]
  \[
  \frac{1}{T} \int_{x} |c_k| dt \leq \frac{1}{T} \int_{x} \sum_{k=-\infty}^{\infty} |c_k| dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} |c_k| < \infty
  \]
  which means that:
  \[
  \frac{1}{T} \int_{x} |x(t)| dt = \frac{1}{T} \int_{x} \left| \sum_{k} c_k e^{jkw_0 t} \right| dt \leq \frac{1}{T} \int_{x} \sum_{k} |c_k e^{jkw_0 t}| dt = \frac{1}{T} \int_{x} \sum_{k=-\infty}^{\infty} |c_k| dt
  \]
Notice that we have said that in order to invert the integral and the series it is sufficient (by Fubini)

$$\sum |k| < \infty$$

which implies

$$\frac{1}{T} \int_{-T/2}^{T/2} |x(t)| dt < \infty$$

absolute integrability over a period.

is this latter condition enough to guarantee the exchange and that the series converges? In general no.

We have to add two more conditions:

**DIRICHLET CONDITIONS**

Any periodic signal $x(t)$ has a Fourier series which converges to $x(t)$ for all $t$ except where $x(t)$ is discontinuous where it converges to the average of the two values at either side of the discontinuity if:

1) $x(t)$ is absolutely integrable over a period
2) $x(t)$ has a finite number of discontinuities over any finite interval
3) $x(t)$ has a finite number of max and min over any finite interval.

However, we have another condition which implies Dirichlet condition 4) and is a finite energy condition which guarantees "weak convergence" of the series.
Let's look again at condition (4) which is the most important.

We have:

\[ \frac{1}{T} \left| \int_{T} |x(t)| dt \right|^2 \leq \frac{1}{T} \int_{T} |x(t)|^2 dt \]

so if

\[ \frac{1}{T} \int_{T} |x(t)|^2 dt < \infty \]

this implies

\[ \frac{1}{T} \left| \int_{T} |x(t)| dt \right| < \infty \]

This is the Dirichlet condition (4).

If only the finite power (energy) conditions is satisfied, we have that the Fourier series converges in Energy to \( x(t) \), although it might not converge in every point:

\[ \left| x(t) - \sum_{k=-\infty}^{+\infty} c_k e^{jkwot} \right| \]

might not be 0 for all \( t \) but:

\[ \lim_{M \to \infty} \int_{T} \left| x(t) - \sum_{k=-M}^{M} c_k e^{jkwot} \right|^2 dt = 0 \]

The energies of the two signals are the same at every point.
Let any signal $x(t)$

Define:  
\[ |x^*(t)|^2 \]  the instantaneous power at time $t$

\[ \frac{1}{T} \int_0^T |x(t)|^2 \, dt \]  the energy of the signal over interval $T$

\[ \frac{1}{T} \int_0^T |x(t)\|^2 \, dt \]  the average power over the interval $T$